

GEOMETRIC SPACE-FREQUENCY ANALYSIS ON MANIFOLDS

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ABSTRACT. This paper gives a survey of methods for the construction of space-frequency concentrated frames on Riemannian manifolds with bounded curvature, and the applications of these frames to the analysis of function spaces. In this general context, the notion of frequency is defined using the spectrum of a distinguished differential operator on the manifold, typically the Laplace-Beltrami operator. Our exposition starts with the case of the real line, which serves as motivation and blueprint for the material in the subsequent sections.

After the discussion of the real line, our presentation starts out in the most abstract setting proving rather general sampling-type results for appropriately defined Paley-Wiener vectors in Hilbert spaces. These results allow a handy construction of Paley-Wiener frames in $L_2(\mathbf{M})$, for a Riemann manifold of bounded geometry, essentially by taking a partition of unity in frequency domain. The discretization of the associated integral kernels then gives rise to frames consisting of smooth functions in $L_2(\mathbf{M})$, with fast decay in space and frequency. These frames are used to introduce new norms in corresponding Besov spaces on \mathbf{M} .

For compact Riemannian manifolds the theory extends to L_p and Besov spaces. Moreover, for compact homogeneous manifolds, one obtains the so-called product property for eigenfunctions of certain operators and proves a cubature formulae with positive coefficients which allow to construct Parseval frames that characterize Besov spaces in terms of coefficient decay.

Throughout the paper, the general theory is exemplified with the help of various concrete and relevant examples, such as the unit sphere and the Poincaré half plane.

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1. INTRODUCTION

In 2004 (see [39]) H. Feichtinger and I. Pesenson wrote: *"It is our strong belief that there exist many real life problems in Signal Analysis and Information Theory which would require non-Euclidean models. A Theory which will unify the ideas of the Classical Sampling Theorem, one of the most beautiful and applied results of the Euclidean Fourier Analysis, with ideas of Differential Geometry and non-Euclidean Harmonic Analysis would be of great interest and importance. We consider the present paper as a foundation for future papers in which we are planning to investigate in details more specific examples such as: 1) Spheres, projective spaces and general compact manifolds. 2) Hyperboloids and general non-compact symmetric spaces. 3) Various Lie groups."*

The purpose of this survey is to provide an introduction to emerging theories of Shannon-type sampling and space-frequency localized frames in various non-Euclidean settings.

We report on the Shannon sampling theory, approximation theory, space-frequency localized frames, and Besov spaces on compact and non-compact manifolds which were developed in [39], [40], [57]-[59], [94]-[134]. These topics are not only of a theoretical interest. Many important applications of multiresolution analysis on manifolds were developed for imaging, geodesy, cosmology, crystallography, scattering theory, biology, and statistics (see [4], [5], [9], [10], [23], [31], [35], [50], [51], [56], [67]-[69], [71], [79]-[81], [135]).

We begin with introducing a rather general, spectral-theoretic setup that allows to prove Shannon-type sampling theorems in abstract Hilbert spaces, as well as the definition and characterization of Besov-type spaces, in a unified language. We then use these results to study sampling theorems, the construction of Paley-Wiener (bandlimited) frames and the characterization of function spaces on Riemannian manifolds (see [104], [105], [113], [118]). The approach works for rather wide classes of Riemannian manifolds, such as general compact manifolds without boundary, bounded domains with smooth boundaries in Euclidean spaces, or non-compact Riemannian manifolds of bounded geometry whose Ricci curvature is bounded from below.

For *compact* Riemannian manifolds we prove generalizations of the Bernstein, Bernstein-Nikolskii and Jackson inequalities. In the case of compact manifolds we go beyond the purely Hilbert space theoretic setting, and include L_p -spaces in the discussion as well, for $1 \leq p \leq \infty$. This allows to characterize elements of the Besov spaces $\mathcal{B}_{p,q}^\alpha(\mathbf{M})$ in terms of approximations by eigenfunctions of elliptic differential operators on \mathbf{M} . For the case of a compact *homogeneous* manifold \mathbf{M} we further sharpen the Bernstein and Bernstein-Nikolskii inequalities, using global derivatives with respect to specific vector fields on \mathbf{M} ; here the above-mentioned elliptic differential operator is the Casimir operator \mathcal{L} . Furthermore, we construct Parseval bandlimited and localized frames in $L_2(\mathbf{M})$ for this setting, and show that they serve to characterize Besov spaces via coefficient decay.

1.1. Overview of the paper. In section 2 we discuss three ways of constructing Paley-Wiener-Schwartz frames in $L_2(\mathbb{R})$. In 2.1 we are using the Fourier transform to introduce functions of the non-negative square root $\sqrt{-d^2/dx^2}$ in the space

$L_2(\mathbb{R})$. In 2.2 we use these results to construct what we call nearly Parseval Paley-Wiener-Schwartz frames in $L_2(\mathbb{R})$ which are comprised of functions which are bandlimited and have fast decay at infinity. In subsection 2.3 we explore the classical Sampling Theorem to construct Parseval Paley-Wiener-Schwartz frames. In subsection 2.4 we establish a cubature formula with positive coefficients for functions in Paley-Wiener space. By means of such formulas and the fact that product of two Paley-Wiener functions is another Paley-Wiener function we develop a third method of constructing Parseval Paley-Wiener-Schwartz frames in $L_2(\mathbb{R})$.

It is the objective of the present article to show how the ideas and methods illustrated in 2 can be extended to Riemannian manifolds.

In our paper we make systematic use of the Spectral Theorem for self-adjoint operators as a substitute for the classical Fourier transform. The approach is motivated by examples described in section 2. However, it should be noted that although the regular Fourier transform considered in section 2 provides spectral resolution for the operator of the first derivative $-id/dx$ in $L_2(\mathbb{R})$ but it is not a spectral resolution for the operator $-d^2/dx^2$ and its non-negative root $\sqrt{-d^2/dx^2}$ in $L_2(\mathbb{R})$.

In Section 3 we introduce a notion of Paley-Wiener (bandlimited) vectors in a Hilbert space \mathcal{H} which is equipped with a self-adjoint operator D , and develop a Shannon-type sampling of such vectors. By constructing appropriate projections of \mathcal{H} onto subspaces of ω -Paley-Wiener vectors $\mathbf{PW}_\omega(D)$, $\omega > 0$ we construct bandlimited frames in \mathcal{H} . We define Besov spaces as interpolation spaces between \mathcal{H} and domains of D^k , $k \in \mathbb{N}$, and show that they can be described in terms of frame coefficients.

Our approach in [105] was to treat a set of "samples" of a vector $f \in \mathcal{H}$ as a set of values $\psi_\nu(f)$ for a specific "sampling" family of functionals ψ_ν for which Plancherel-Polya-type inequalities (=frame inequalities) hold on Paley-Wiener subspaces. The Spectral Theorem allows to decompose every vector in \mathcal{H} into a series of Paley-Wiener vectors. Then an application of our Sampling Theorem 3.8 leads to the construction of Paley-Wiener frames for \mathcal{H} (Theorem 3.9).

In Subsection 3.6 we formulate and prove an important result (Theorem 3.14) about interpolation and approximation spaces. This result is essentially due to Peetre-Sparr [93] and Butzer-Scherer [17], but we formulate and prove it in a form which is most suitable for our purposes (see also [73]). In particular, our formulation is more general than a similar Theorem 9.1 in Ch. 7 in [29]. In Subsection 3.7 we describe abstract Besov subspaces in terms of approximations by Paley-Wiener vectors and in terms of coefficients with respect to our Paley-Wiener frames.

When it comes to the space $\mathcal{H} = L_2(\mathbf{M})$, where \mathbf{M} is a manifold, the families of "sampling" functionals $\{\psi_\nu\}$ are just families of compactly supported distributions (with small supports) associated with what we call metric lattices of points $\{x_k\}$ on \mathbf{M} . This term is used to emphasize that the points $\{x_k\}$ are distributed over \mathbf{M} "almost uniformly" and that they are separated.

Not every metric space possesses lattices of points with the properties we need for our sampling theory. In Subsection 4.1 we clarify this issue. It was shown in [108] that if a Riemannian manifold has bounded geometry and its Ricci curvature is bounded from below then one can construct a sequence of lattices whose mesh radius tends to zero. Note that the property of bounded geometry is essentially equivalent to the fact that all covariant derivatives of the Riemann curvature are bounded from above. This shows that our conditions are rather natural, since

appropriate uniformly distributed and separated sets of points can exist only if the curvature (in one sense or another) is bounded from above and from below. The rest of the Section 4 is devoted to descriptions of manifolds of common interest.

In Section 5 we implement the general scheme of Section 2 in the spaces $L_2(\mathbf{M})$, for manifolds \mathbf{M} satisfying the assumptions of Section 3. Following the general scheme of Section 3, the central tool for sampling theory are Poincaré-type estimates, which we derive for Riemannian manifolds in Section 4. To construct frames which are almost tight we use the so-called *average* sampling in a way that includes and generalizes the pointwise sampling. In the case of a straight line average sampling was considered for the first time in [36]. In the case of manifolds average sampling was developed in [108], [111]. We do not discuss reconstruction algorithms in detail, but it can be done by the following methods (besides using dual frames): (1) reconstruction by using variational splines on manifolds [101]-[118]; (2) reconstruction using iterations [39], [40]; (3) the frame algorithm [61].

In Sections 6 we introduce and analyze kernels associated to elliptic differential operators on general compact Riemannian manifolds. The most important result here concerns the localization of such kernels (Theorem 6.1) which implies an analogue of the Littlewood-Paley decomposition of functions in the spaces $L_p(\mathbf{M})$, $1 \leq p \leq \infty$, on compact Riemannian manifolds (Theorem 7.4). One can find other approaches to the Littlewood-Paley decompositions on manifolds (see for example [71], [141], [142]).

Section 8 is devoted to Parseval space-frequency localized frames on compact homogeneous manifolds. The main result here is Theorem 8.6 which was proved in [57]. This Theorem is based on two non-trivial facts: the product of eigenfunctions of certain elliptic differential operators (Theorem 8.1) and on a cubature formula with positive coefficients which allows for exact integration of respected eigenfunctions (Theorem 8.5). Let us just mention that a set of similar facts holds true for sub-Laplacians on compact homogeneous manifolds [133].

In Section 9 an approximation theory by eigenfunctions of elliptic operators on compact manifolds is developed. These results lead to a characterization of Besov spaces in terms of sampling (Theorem 9.7). In Section 10 we discuss approximation theory on compact homogeneous manifolds. In particular, a mixed modulus of continuity is introduced and Besov spaces are characterized in terms of this modulus of continuity [94]-[96]. Section 10.3 contains characterization of Besov spaces in terms of frame coefficients. Similar theorems can be proved in the case of a sub-Laplacian and sub-elliptic spaces on compact homogeneous manifolds [133].

Here is a very brief account of related work, mostly by other authors. The papers [3], [8], [15], [23]-[27], [41], [45], [47], [52], [62], [63], [70], [77], [84], [85], [86], [148], contain a number of results about frames, wavelets, and Besov spaces on Riemannian manifolds, on Lie groups of polynomial growth, on metric-measure spaces, and on quasi-metric measure spaces. One can say that most of these papers generalize and further develop ideas which are rooted in the classical Littlewood-Paley theory and/or Calderon reproducing formula.

In particular, it is well understood by now that a productive generalization of the Littlewood-Paley theory should be based on a decomposition of identity operator into a series of kernel operators with appropriately localized kernels. It was proved in [62], [63] that any reasonably nice metric measure space admits such decomposition.

Among the papers devoted to metric-measure and quasi-metric measure spaces the articles [23], [24], [41], [70], [77], [85] are the closest to our approach, since they also aim to construct space-frequency localized frames. To incorporate a notion of frequency into a setting of quasi-metric measure spaces, the authors of these papers impose some additional conditions. They, essentially, assume the existence of a self-adjoint operator in a corresponding L_2 space whose heat semigroup is a kernel operator with a kernel obeying estimates resembling the heat kernel estimates on Euclidean space. This way they are able to define a notion of bandlimitedness and to construct space-frequency localized frames. The most advanced results in such setting were recently obtained in [24].

It should be noted that the most interesting situations where one finds the conditions of these papers satisfied are still manifolds: compact Riemannian manifolds, non-compact manifolds with curvature bounded from below, groups of polynomial growths and their homogeneous manifolds.

Again, our goal is to give a concise introduction to the fast developing subject of sampling and wavelet-like frames on manifolds by a description of a few underlying ideas, which provide a basis for discoveries in [39], [40], [57]-[59], [94]-[134].

2. BASIC EXAMPLE: PALEY-WIENER-SCHWARTZ FRAMES IN $L_2(\mathbb{R})$

2.1. Smooth decomposition of $L_2(\mathbb{R})$ into Paley-Wiener subspaces. We take the first-order pseudo-differential operator $D = \sqrt{-d^2/dx^2}$ as the positive square root of the positive operator $-d^2/dx^2$. If F belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$ then following the spirit of the Spectral Theorem (see (3.2)) one can introduce the operator $F(D)$ by the formula

$$(2.1) \quad F\left(\sqrt{-d^2/dx^2}\right) f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\lambda} F(\lambda) \hat{f}(\lambda) d\lambda, \quad f \in \mathcal{S}(\mathbb{R}),$$

where the Fourier transform \hat{f} is defined as

$$\hat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\lambda} f(x) dx, \quad f \in \mathcal{S}(\mathbb{R}).$$

The operator $F\left(\sqrt{-d^2/dx^2}\right)$ is convolution with the Schwartz function $\check{F} \in \mathcal{S}(\mathbb{R})$ which is inverse Fourier transform of F :

$$F\left(\sqrt{-d^2/dx^2}\right) f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \check{F}(x-y) f(y) dy.$$

In particular, for any positive t we have

$$(2.2) \quad F\left(t\sqrt{-d^2/dx^2}\right) f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{t} \check{F}\left(\frac{x-y}{t}\right) f(y) dy = \int_{\mathbb{R}} K_t^F(x, y) f(y) dy,$$

where

$$(2.3) \quad K_t^F(x, y) = \frac{1}{t\sqrt{2\pi}} \check{F}\left(\frac{x-y}{t}\right).$$

Moreover, one clearly has that for any $N > 0$ there exists a constant C_N such that

$$(2.4) \quad |K_t^F(x, y)| \leq \frac{C_N}{t} \left[1 + \frac{|x-y|}{t}\right]^{-N}.$$

Let $g \in C^\infty(\mathbb{R})$ be a non-increasing function such that $\text{supp}(g) \subset [-2, 2]$, and $g(\lambda) = 1$ for $\lambda \in [-1, 1]$, $0 \leq g(\lambda) \leq 1$. We now let

$$h(\lambda) = g(\lambda) - g(2\lambda),$$

which entails $\text{supp}(h) \subset [-2, -2^{-1}] \cup [2^{-1}, 2]$, and use this to define

$$(2.5) \quad F_0(\lambda) = \sqrt{g(\lambda)}, F_j(\lambda) = \sqrt{h(2^{-j}\lambda)}, j \geq 1, j \in \mathbb{N},$$

as well as

$$G_j(\lambda) = [F_j(\lambda)]^2 = F_j^2(\lambda), j \geq 0, j \in \mathbb{N}.$$

As a result of the definitions, we get for all $\lambda \in \mathbb{R}$ the equations

$$(2.6) \quad \sum_{j=0}^m G_j(\lambda) = \sum_{j=0}^m F_j^2(\lambda) = g(2^{-m}\lambda),$$

and as a consequence

$$(2.7) \quad \sum_{j \geq 0} G_j(\lambda) = \sum_{j \geq 0} F_j^2(\lambda) = 1, \quad \lambda \in \mathbb{R},$$

with finitely many nonzero terms occurring in the sums for each fixed λ . One has

$$F_j^2 \left(\sqrt{-d^2/dx^2} \right) f = \mathcal{F}^{-1} \left(F_j^2(\lambda) \mathcal{F}f(\lambda) \right), \quad j \geq 1, j \in \mathbb{N},$$

and thus

$$(2.8) \quad f = \mathcal{F}^{-1} \mathcal{F}f(\lambda) = \mathcal{F}^{-1} \left(\sum_{j \geq 0} F_j^2(\lambda) \mathcal{F}f(\lambda) \right) = \sum_{j \geq 0} F_j^2 \left(\sqrt{-d^2/dx^2} \right) f = \sum_{j \geq 0} G_j \left(\sqrt{-d^2/dx^2} \right) f.$$

Since $F_j \left(\sqrt{-d^2/dx^2} \right)$ is a self-adjoint operator we obtain

$$\begin{aligned} \left\| F_j \left(\sqrt{-d^2/dx^2} \right) f \right\|^2 &= \left\langle F_j \left(\sqrt{-d^2/dx^2} \right) f, \left(\sqrt{-d^2/dx^2} \right) f \right\rangle = \\ &= \left\langle F_j^2 \left(\sqrt{-d^2/dx^2} \right) f, f \right\rangle \end{aligned}$$

and then

$$(2.9) \quad \|f\|^2 = \sum_{j \geq 0} \left\langle F_j^2 \left(\sqrt{-d^2/dx^2} \right) f, f \right\rangle = \sum_{j \geq 0} \left\| F_j \left(\sqrt{-d^2/dx^2} \right) f \right\|^2.$$

Since the functions G_j, F_j , have their supports in $[-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}]$, the functions $F_j^2 \left(\sqrt{-d^2/dx^2} \right) f$ and $G_j \left(\sqrt{-d^2/dx^2} \right) f$ are bandlimited to $[-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}]$, whenever $j \geq 1$, and to $[-2, 2]$ for $j = 0$. They clearly belong to the Schwartz space $\mathcal{S}(\mathbb{R})$.

2.2. A method of constructing almost Parseval Paley-Wiener-Schwartz frames.

Definition 2.1. *The Paley-Wiener space $PW_\omega(\mathbb{R})$, $\omega > 0$, is introduced as the space of all $f \in L_2(\mathbb{R})$ whose L_2 -Fourier transform has support in $[-\omega, \omega]$.*

Using the Fourier transform one can easily verify that a function $f \in L_2(\mathbb{R})$ belongs to the space $PW_\omega(\mathbb{R})$ if and only if the following Bernstein inequality holds

$$\|f^{(r)}\| \leq \omega^r \|f\| \quad r \in \mathbb{N}.$$

For a given $\rho > 0$ and $0 < \epsilon < 1$ consider a sequence $\{x_k\}$ such that

$$(2.10) \quad |x_k - x_{k+1}| \leq \rho, \quad |x_k - x_{k+1}| \geq \rho/(1 + \epsilon)$$

and set $I_k = (x_k, x_{k+1})$. The Fundamental Theorem of calculus and the Holder inequality imply

$$\begin{aligned} \int_{x_k}^x f'(t) dt &= f(x) - f(x_k), \quad x_k \leq x \leq x_{k+1} \\ |f(x) - f(x_k)| &\leq \int_{x_k}^x |f'(t)| dt \leq \rho^{1/2} \left(\int_{I_k} |f'|^2 \right)^{1/2}, \\ |f(x) - f(x_k)|^2 &\leq \rho \int_{I_k} |f'|^2, \end{aligned}$$

and another integration over I_k gives

$$(2.11) \quad \|f - f(x_k)\|_{I_k}^2 \leq \rho^2 \|f'\|_{I_k}^2.$$

For $0 < \alpha < 1$ and any A, B one has

$$(2.12) \quad (1 - \alpha)|A|^2 \leq \frac{1}{\alpha}|A - B|^2 + |B|^2, \quad 0 < \alpha < 1.$$

Using (2.11), (2.12) we obtain

$$(2.13) \quad \left(1 - \frac{\epsilon}{3}\right) \|f\|^2 \leq \sum_k |I_k| |f(x_k)|^2 + \frac{3}{\epsilon} \rho^2 \|f'\|^2.$$

Applying the Bernstein inequality for $f \in PW_\omega(\mathbb{R})$ we get

$$(2.14) \quad \left(1 - \frac{\epsilon}{3}\right) \|f\|^2 \leq \sum_k |I_k| |f(x_k)|^2 + \frac{3}{\epsilon} \rho^2 \omega \|f\|^2.$$

Assuming $\omega > 1$ and choosing

$$\rho \leq \frac{1}{3} \omega^{-1} \epsilon \leq \frac{1}{3} \omega^{-1/2} \epsilon$$

we obtain

$$\frac{3}{\epsilon} \rho^2 \omega \leq \frac{\epsilon}{3}.$$

Finally it gives us

$$(2.15) \quad \left(1 - \frac{2}{3} \epsilon\right) \|f\|^2 \leq \sum_k |I_k| |f(x_k)|^2.$$

We consider the function

$$\xi(x) = \begin{cases} e \exp(1/(|x|^2 - 1)) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Set $\xi_k(x) = \xi((1 + \epsilon)|I_k|^{-1}(x - x_k))$. Since

$$\xi_k(x_{k+1})f(x_{k+1}) = 0, \quad \xi'_k(x) = (1 + \epsilon)|I_k|^{-1}\xi((1 + \epsilon)|I_k|^{-1}(x - x_k)),$$

and $|I_k|^{-1} \leq (1 + \epsilon)\rho^{-1}$ we obtain

$$\begin{aligned} |f(x_k)| &= \left| \int_{x_k}^{x_{k+1}} (\xi_k(x)f(x))' \right| \leq \int_{x_k}^{x_{k+1}} \left| \xi'_k(x)f(x) + \xi_k(x)f'(x) \right| \leq \\ &(1 + \epsilon)^2 \rho^{-1} \int_{x_k}^{x_{k+1}} |f(x)| + \int_{x_k}^{x_{k+1}} |f(x)'| \leq (1 + \epsilon)^2 \rho^{-1/2} \|f\|_{I_k} + \rho^{1/2} \|f'\|_{I_k}. \end{aligned}$$

Then for $f \in PW_\omega(\mathbb{R})$

$$|f(x_k)| \leq \left((1 + \epsilon)^2 \rho^{-1/2} + \omega \rho^{1/2} \right) \|f\|_{I_k}.$$

It gives

$$\sqrt{|I_k|} |f(x_k)| \leq ((1 + \epsilon)^2 + \omega \rho) \|f\|_{I_k}.$$

Because $\rho \leq \frac{1}{3}\omega^{-1}\epsilon$, $0 < \epsilon < 1$, we have

$$(1 + \epsilon)^2 + \omega \rho \leq 1 + \frac{10}{3}\epsilon.$$

This leads to

$$\sqrt{|I_k|} |f(x_k)| \leq \left(1 + \frac{10}{3}\epsilon \right) \|f\|_{I_k},$$

and

$$(2.16) \quad \sum_k |I_k| |f(x_k)|^2 \leq \left(1 + \frac{10}{3}\epsilon \right)^2 \|f\|^2.$$

Using the same notations as above we can now formulate the following theorem about irregular sampling.

Theorem 2.2. *If $0 < \epsilon < 1$ and $0 < \rho \leq \frac{\epsilon}{3}\omega^{-1}$ then Plancherel-Polya inequalities hold*

$$(2.17) \quad \left(1 - \frac{2}{3}\epsilon \right) \|f\|^2 \leq \sum_k |I_k| |f(x_k)|^2 \leq \left(1 + \frac{10}{3}\epsilon \right)^2 \|f\|^2, \quad f \in PW_\omega(\mathbb{R}).$$

See [6], [12], [30], [136], [137] for the classical Plancherel-Polya inequalities.

Note, that if δ_{x_k} is a Dirac distribution where $\{x_k\}$ are defined in (2.10) then the Plancherel-Polya inequalities (2.17) mean that projections of $\left\{ \sqrt{|I_k|} \delta_{x_k} \right\}$ onto $PW_\omega(\mathbb{R})$ form a frame in this space.

We return to notations of section 2.1. Thus, the operator $F_j \left(\sqrt{-d^2/dx^2} \right)$ is a projector of $L_2(\mathbb{R})$ into $PW_{2j+1}(\mathbb{R})$. For a fixed $0 < \epsilon < 1$ pick a ρ such that $0 < \rho \leq \frac{\epsilon}{3}\omega^{-1}$. Let $\{I_k\}$ be a corresponding partition $I_k = (x_k, x_{k+1})$ considered in (2.10).

In the sense of distributions one has

$$\left(F_j \left(\sqrt{-d^2/dx^2} \right) \sqrt{|I_k|} \delta_{x_k} \right) (x) = \left(\mathcal{F}^{-1} \left(F_j \sqrt{|I_k|} e^{ix_k \cdot} \right) \right) (x),$$

where the formula $\mathcal{F} \delta_{x_k} = e^{ix_k \xi}$ was used. At the same time, according to (2.2)

$$\left(F_j \left(\sqrt{-d^2/dx^2} \right) \sqrt{|I_k|} \delta_{x_k} \right) (x) = \int_{\mathbb{R}} K_1^{F_j}(x, y) \sqrt{|I_k|} \delta_{x_k}(y) dy =$$

$$(2.18) \quad K_1^{F_j}(x_k, y) = \frac{1}{\sqrt{2\pi}} \tilde{F}_j(x_k - y) = \varphi_{x_k}^j(y) = \varphi_k^j(y).$$

It is obvious that every φ_k^j belongs to $PW_{2^{j+1}}(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$ which means that it is perfectly localized on the frequency side and "essentially" localized in space(time).

Since the operator $F_j(\sqrt{-d^2/dx^2})$ is self-adjoint one has

$$\left\langle F_j(\sqrt{-d^2/dx^2}) f, \sqrt{|I_k|} \delta_{x_k} \right\rangle = \left\langle f, \varphi_k^j \right\rangle.$$

Combining (2.9) and (2.17) we obtain that the following frame inequalities hold

$$(2.19) \quad \left(1 - \frac{2}{3}\epsilon\right) \|f\|^2 \leq \sum_j \sum_{k \in \mathbb{Z}} \left| \left\langle f, \varphi_k^j \right\rangle \right|^2 \leq \left(1 + \frac{10}{3}\epsilon\right)^2 \|f\|^2, \quad f \in L_2(\mathbb{R}).$$

The classical result of Duffin and Schaeffer [30] says that the inequalities (2.19) imply the existence of a *dual frame* $\{\Phi_k^j\}$ such that any function $f \in PW_{2^{j+1}}(\mathbb{R})$ can be reconstructed according to the following formula

$$(2.20) \quad f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left\langle f, \varphi_k^j \right\rangle \Phi_k^j(x).$$

It is clear that each Φ_k^j belongs to $PW_{2^{j+1}}(\mathbb{R})$.

2.3. Constructing Parseval Paley-Wiener-Schwartz frames using Classical Sampling Theorem. The classical sampling theorem says, that if f is ω -bandlimited then f is completely determined by its values at points $k\pi/\omega, k \in \mathbb{Z}$, and can be reconstructed in a stable way from the samples $f(k\pi/\omega)$, i.e.

$$(2.21) \quad f(x) = \sum_{k \in \mathbb{Z}} f(k\pi/\omega) \frac{\sin(\omega(x - k\pi/\omega))}{\omega(x - k\pi/\omega)},$$

where convergence is understood in the L_2 -sense. Moreover, the following equality between "continuous" and "discrete" norms holds true

$$(2.22) \quad \left(\int_{-\infty}^{+\infty} |f(x)|^2 dx \right)^{1/2} = \left(\sum_{j \in \mathbb{Z}} \frac{1}{\omega} |f(k\pi/\omega)|^2 \right)^{1/2}.$$

This equality follows from the fact that the functions $e^{2\pi i t(k\pi/\omega)}$ form an orthonormal basis in $L_2[-\omega, \omega]$. Now we take $\omega = 2^{j+1}$. Since the operator $F_j(\sqrt{-d^2/dx^2})$ is self-adjoint one has

$$\left\langle F_j(\sqrt{-d^2/dx^2}) 2^{(-j-1)/2} \delta_{x_{k\pi 2^{-j-1}}} \right\rangle = \left\langle f, F_j(\sqrt{-d^2/dx^2}) 2^{(-j-1)/2} \delta_{x_{k\pi 2^{-j-1}}} \right\rangle$$

and formulas (2.9) and (2.22) imply that the set of functions

$$\psi_k^j(x) = \left(F_j(\sqrt{-d^2/dx^2}) 2^{(-j-1)/2} \delta_{x_{k\pi 2^{-j-1}}} \right)(x)$$

is a Parseval frame in $L_2(\mathbb{R})$. For the same reasons as above we see that $\psi_k^j \in PW_{2^{j+1}}(\mathbb{R}) \cap \mathcal{S}(\mathbb{R})$. Moreover, the general frame theory implies that the following reconstruction formula holds

$$(2.23) \quad f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \left\langle f, \psi_k^j \right\rangle \psi_k^j(x).$$

2.4. Constructing Parseval Paley-Wiener-Schwartz frames using a cubature formula. The following result will be used in this section (compare to a similar statement in [57]).

Theorem 2.3. *If $0 < \gamma < 1$, $\rho < \frac{1}{6}\omega^{-1}\gamma$ and $\{x_k\}$ is such that $\rho/2 \leq |x_k - x_{k+1}| \leq \rho$ then there exist strictly positive coefficients $\lambda_{x_k} > 0$, which are of the order ρ , for which the following equality holds for all functions in $PW_\omega(\mathbb{R}) \cap L_1(\mathbb{R})$:*

$$(2.24) \quad \int_{\mathbb{R}} f dx = \sum_{x_k} \lambda_{x_k} f(x_k).$$

Proof. By using Riemann sums and the Fundamental Theorem of calculus we obtain for functions in the Schwartz space the following inequalities

$$(2.25) \quad \left| \int_{\mathbb{R}} f(x) dx - \sum_{x_k} f(x_k) |I_k| \right| = \left| \sum_k \int_{I_k} f(x) dx - \sum_{x_k} \int_{I_k} f(x_k) dx \right| \leq \sum_k \int_{I_k} |f(x) - f(x_k)| dx \leq \rho^{1/2} \|f'\|$$

Thus, for $f \in PW_\omega(\mathbb{R})$ using the Bernstein inequality and the left side of (2.17)

$$\left| \int_{\mathbb{R}} f(x) dx - \sum_{x_k} f(x_k) |I_k| \right| \leq \rho^{3/2} \|f'\| \leq (1 - \gamma)^{-1/2} \rho^2 \omega \left(\sum_k |f(x_k)|^2 \right)^{1/2}.$$

Consider the sampling operator

$$S : f \rightarrow \{f(x_k)\},$$

which maps $PW_\omega(\mathbb{R})$ onto a V which is a subspace of the space ℓ^2 with its standard norm.

If $u \in V$, denote the linear functional $y \rightarrow (y, u)$ on V by ℓ_u . By our Plancherel-Polya inequalities (2.17), the map

$$\{f(x_k)\} \rightarrow \int_{\mathbb{R}} f dx$$

is a well-defined linear functional on the closed Hilbert space V , and so equals ℓ_v for some $v \in V$, which may or may not have all components positive. On the other hand, if w is the vector with components $\{|I_k|\}$, then w might not be in V , but it has all components positive and of the right size

$$|I_k| \sim \rho.$$

Since, for any vector $u \in V$ the norm of u is exactly the norm of the corresponding functional ℓ_u , the above inequality tells us that

$$(2.26) \quad \|Pw - v\| \leq \|w - v\| \leq (1 - \gamma)^{-1/2} \rho^2 \omega,$$

where P is the orthogonal projection onto V . Accordingly, if z is the vector $v - Pw$, then

$$(2.27) \quad v + (I - P)w = w + z,$$

where $\|z\| \leq (1 - \gamma)^{-1/2} \rho^2 \omega$. Note, that all components of the vector w are of order $O(\rho)$, while the order of $\|z\|$ is $O(\rho^2)$. Thus if $\rho\omega$ is sufficiently small, then $\lambda := w + z$ has all components positive and of the right size. Since $\lambda = v + (I - P)w$,

the linear functional $y \rightarrow (y, \lambda)$ on V equals ℓ_v . In other words, if the vector λ has components $\{\lambda_{x_k}\}$, then

$$\sum_{x_k} f(x_k) \lambda_{x_k} = \int_{\mathbb{R}} f dx$$

for all $f \in PW_{\omega}(\mathbb{R})$, as desired. \square

The following statement immediately follows from the fact that the Fourier transform of a product of two functions in $L_2(\mathbb{R})$ is a convolution of their Fourier transforms.

Lemma 2.4. *If $f, g \in PW_{\omega}(\mathbb{R})$ then their product fg is in $PW_{2\omega}(\mathbb{R})$. In particular, if $f \in PW_{\omega}(\mathbb{R})$ then $|f|^2 = f\bar{f}$ belongs to $PW_{2\omega}(\mathbb{R})$.*

Thus we can use the above quadrature rule for $F_j \left(\sqrt{-d^2/dx^2} \right) f = f_j$

$$\|f_j\|^2 = \int_{\mathbb{R}} |f_j|^2 dx = \sum_{x_k \in M_{\rho}} \lambda_{x_k} |f_j(x_k)|^2.$$

Using (2.9) and introducing functions

$$\theta_k^j = \sqrt{\lambda_{x_k}} F_j \left(\sqrt{-d^2/dx^2} \right) \delta_{x_k}$$

we obtain a Parseval frame in $L_2(\mathbb{R})$ since

$$\|f\|^2 = \sum_j \sum_k \left| \langle f_j, \theta_k^j \rangle \right|^2.$$

This fact implies that the following reconstruction formula holds

$$f = \sum_j \sum_k \langle f_j, \theta_k^j \rangle \theta_k^j.$$

To summarize these examples we list the following crucial facts which were used in the previous constructions:

- (1) the Fourier transform provides the Spectral Resolution of the operator $-d/dx$ in $L_2(\mathbb{R})$;
- (2) existence of the irregular and regular sampling theorems;
- (3) the fact that the product of two functions in $PW_{\omega}(\mathbb{R})$ is a function in $PW_{2\omega}(\mathbb{R})$;
- (4) existence of exact quadrature formula with positive coefficients for Paley-Wiener function.

The goal of our survey is to demonstrate that:

- (1) the method developed in section 2.2 can be extended to general Hilbert spaces [105], to compact Riemannian manifolds [107], [109], to non-compact manifolds of bounded geometry whose Ricchi curvature is bounded from below [104], [108], to non-compact symmetric Riemannian manifolds [111], [125], to domains in \mathbb{R}^n [132];
- (2) the method developed in section 2.4 can be extended to homogeneous compact Riemannian manifolds [57] and homogeneous manifolds with sub-elliptic structure [133].

We note that the method of section 2.3 can not be extended to Riemannian manifolds due, in particular, to the lack of uniformly spaced sets of points.

It should be also mentioned that the methods of 2.2 were extended to metric(quantum) and combinatorial graphs [48], [110], [112], [115], [121], [123].

3. SHANNON SAMPLING, PALEY-WIENER FRAMES AND ABSTRACT BESOV SUBSPACES

3.1. Paley-Wiener vectors in Hilbert spaces. Consider a self-adjoint positive definite operator L in a Hilbert space \mathcal{H} . Let \sqrt{L} be the positive square root of L . According to the spectral theory for such operators [11] there exists a direct integral of Hilbert spaces $X = \int X(\lambda) dm(\lambda)$ and a unitary operator \mathcal{F} from \mathcal{H} onto X , which transforms the domains of $L^{k/2}$, $k \in \mathbb{N}$, onto the sets $X_k = \{x \in X | \lambda^k x \in X\}$ with the norm

$$(3.1) \quad \|x(\lambda)\|_{X_k} = \langle x(\lambda), x(\lambda) \rangle_{X(\lambda)}^{1/2} = \left(\int_0^\infty \lambda^{2k} \|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda) \right)^{1/2}.$$

and satisfies the identity $\mathcal{F}(L^{k/2}f)(\lambda) = \lambda^k(\mathcal{F}f)(\lambda)$, if f belongs to the domain of $L^{k/2}$. We call the operator \mathcal{F} the Spectral Fourier Transform [97], [104]. As known, X is the set of all m -measurable functions $\lambda \mapsto x(\lambda) \in X(\lambda)$, for which the following norm is finite:

$$\|x\|_X = \left(\int_0^\infty \|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda) \right)^{1/2}$$

For a function F on $[0, \infty)$ which is bounded and measurable with respect to dm one can introduce the operator $F(\sqrt{L})$ by using the formula

$$(3.2) \quad F(\sqrt{L})f = \mathcal{F}^{-1}F(\lambda)\mathcal{F}f, \quad f \in \mathcal{H}.$$

If F is real-valued the operator $F(\sqrt{L})$ is self-adjoint.

Remark 3.1. We start with an operator L and switch to \sqrt{L} because in many applications (see below) a second-order differential operator L appears first, but it is more natural to work with a first order pseudo-differential operator \sqrt{L} . This will become apparent when we discuss sampling density in Subsection 5.3.

Definition 3.2. For \sqrt{L} as above we will say that a vector $f \in \mathcal{H}$ belongs to the Paley-Wiener space $\mathbf{PW}_\omega(\sqrt{L})$ if the support of the Spectral Fourier Transform $\mathcal{F}f$ is contained in $[0, \omega]$.

The next two facts are obvious.

Theorem 3.3. The spaces $\mathbf{PW}_\omega(\sqrt{L})$ have the following properties:

- (1) the space $\mathbf{PW}_\omega(\sqrt{L})$ is a linear closed subspace in \mathcal{H} .
- (2) the space $\bigcup_{\omega>0} \mathbf{PW}_\omega(\sqrt{L})$ is dense in \mathcal{H} ;

Next we denote by \mathcal{H}^k the domain of $L^{k/2}$. It is a Banach space, equipped with the graph norm $\|f\|_k = \|f\| + \|L^{k/2}f\|$. The next theorem contains generalizations of several results from classical harmonic analysis (in particular the Paley-Wiener theorem). It follows from our results in [104] and [117].

Theorem 3.4. The following statements hold:

- (1) (Bernstein inequality) $f \in \mathbf{PW}_\omega(\sqrt{L})$ if and only if $f \in \mathcal{H}^\infty = \bigcap_{k=1}^\infty \mathcal{H}^k$, and the following Bernstein inequalities holds true

$$(3.3) \quad \|L^{s/2}f\| \leq \omega^s \|f\| \quad \text{for all } s \in \mathbb{R}_+;$$

- (2) (Paley-Wiener theorem) $f \in \mathbf{PW}_\omega(\sqrt{L})$ if and only if for every $g \in \mathcal{H}$ the scalar-valued function of the real variable $t \mapsto \langle e^{it\sqrt{L}}f, g \rangle$ is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type ω ;

- (3) (Riesz-Boas interpolation formula) $f \in \mathbf{PW}_\omega(\sqrt{L})$ if and only if $f \in \mathcal{H}^\infty$ and the following Riesz-Boas interpolation formula holds for all $\omega > 0$:

$$(3.4) \quad i\sqrt{L}f = \frac{\omega}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k-1}}{(k-1/2)^2} e^{i(\frac{\pi}{\omega}(k-1/2))\sqrt{L}} f.$$

Proof. (1) follows immediately from the definition and representation (3.1). To prove (2) it is sufficient to apply the classical Bernstein inequality [87] for the uniform norm on \mathbb{R} to every function $\langle e^{it\sqrt{L}}f, g \rangle$, $g \in \mathcal{H}$. To prove (3) one has to apply the classical Riesz interpolation formula on \mathbb{R} [87] to the functions $\langle e^{it\sqrt{L}}f, g \rangle$. \square

Remark 3.5. For trigonometric polynomials in $L_p(\mathbb{T})$, $1 \leq p \leq \infty$, and the operator $\frac{d}{dt}$, the identity (3.4) was proved by Riesz [138]. For entire functions of exponential type in $L_p(\mathbb{R})$, $1 \leq p \leq \infty$, and $\frac{d}{dt}$ this identity was proved by Boas [12].

3.2. Frames in Hilbert spaces. A family of vectors $\{\theta_v\}$ in a Hilbert space \mathcal{H} is called a frame if there exist constants $A, B > 0$ such that

$$(3.5) \quad A\|f\|^2 \leq \sum_v |\langle f, \theta_v \rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in \mathcal{H}.$$

The largest A and smallest B are called lower and upper frame bounds.

The family of scalars $\{\langle f, \theta_v \rangle\}$ represents a set of measurements of a vector f . In order to resynthesize the vector f from this collection of measurements in a linear way one has to find another (dual) frame $\{\Theta_v\}$. Then a reconstruction formula is

$$(3.6) \quad f = \sum_v \langle f, \theta_v \rangle \Theta_v.$$

Dual frames are not unique in general. Moreover it may be difficult to find a dual frame in concrete situations.

If in particular $A = B = 1$ the frame is said to be tight or Parseval. Parseval frames are similar in many respects to orthonormal wavelet bases. For example, if in addition all vectors θ_v are unit vectors, then the frame is an orthonormal basis.

The main feature of Parseval frames is that decomposing and synthesizing a vector from known data are tasks carried out with the same family of functions, i.e., the Parseval frame is its own dual frame. The important differences between frames and, say, orthonormal bases is their redundancy that helps reduce for example the effect of noise in data.

Frames in Hilbert spaces of functions whose members have simultaneous localization in space and frequency arise naturally in wavelet analysis on Euclidean spaces, when the continuous wavelet transforms are discretized.

3.3. Sampling in abstract Paley-Wiener spaces. We assume that there exist a $C > 0$ and $m_0 \geq 0$ such that for any $0 < \rho < 1$ there exists a set of functionals $\mathcal{A}^{(\rho)} = \{\mathcal{A}_k^{(\rho)}\}$, defined on \mathcal{H}^{m_0} , for which

$$(3.7) \quad \|f\|^2 \leq C \left(\sum_k \left| \mathcal{A}_k^{(\rho)}(f) \right|^2 + \rho^{2m} \|L^{m/2} f\|^2 \right), \quad f \in \mathcal{H}^m, \quad m > m_0,$$

and

$$(3.8) \quad \sum_k \left| \mathcal{A}_k^{(\rho)}(f) \right|^2 \leq C \|f\|^2, \quad \text{for all } f \in \mathcal{H}^m, \quad m > m_0.$$

Remark 3.6. In notations of section 2.2 if

$$\mathcal{A}_k^{(\rho)}(f) = \sqrt{|I_k|} |f(x_k)|, \quad |I_k| \leq \rho,$$

then inequality (3.7) is similar to (2.13) and (3.8) is similar to (2.16).

Remark 3.7. Following [105], [108] we call inequality (3.7) a Poincaré-type inequality since it is an estimate of the norm of f through the norm of its “derivative” $L^{m/2} f$.

Let us introduce vectors $\mu_k \in \mathcal{H}$ such that

$$\langle f, \mu_k \rangle = \mathcal{A}_k^{(\rho)}(f), \quad f \in \mathcal{H}^m, \quad m > m_0.$$

Let \mathcal{P}_ω be the orthogonal projection of \mathcal{H} onto $\mathbf{PW}_\omega(\sqrt{L})$ and put

$$(3.9) \quad \phi_k^\omega = \mathcal{P}_\omega \mu_k.$$

Using the Bernstein inequality (3.3) we obtain the following statement.

Theorem 3.8. (*Sampling Theorem*)

Assume that assumptions (3.7) and (3.8) are satisfied and for a given $\omega > 0$ and $\delta \in (0, 1)$ pick a ρ such that

$$\rho^{2m} = C^{-1} \omega^{-2m} \delta.$$

Then the family of vectors $\{\phi_k^\omega\}$ in (3.9) is a frame for the Hilbert space $\mathbf{PW}_\omega(\sqrt{L})$ and

$$(3.10) \quad (1 - \delta) \|f\|^2 \leq \sum_k |\langle f, \phi_k^\omega \rangle|^2 \leq \|f\|^2, \quad f \in \mathbf{PW}_\omega(\sqrt{L}).$$

The canonical dual frame $\{\Theta_k^\omega\}$ has the property $\Theta_k^\omega \in \mathbf{PW}_\omega(\sqrt{L})$ and provides the following reconstruction formulas

$$(3.11) \quad f = \sum_k \langle f, \phi_k^\omega \rangle \Theta_k^\omega = \sum_k \langle f, \Theta_k^\omega \rangle \phi_k^\omega, \quad f \in \mathbf{PW}_\omega(\sqrt{L}).$$

3.4. Partitions of unity on the frequency side. The construction of frequency-localized frames is typically achieved via spectral calculus. The idea is to start from a partition of unity on the positive real axis. In the following, we will be considering two different types of such partitions, whose construction we now describe in some detail.

Now we are going to construct partitions of unity F_j and $G_j = F_j^2$ which are similar to one that were introduced in (2.5)-(2.7).

Let $g \in C^\infty(\mathbb{R}_+)$ be a non-increasing function such that $\text{supp}(g) \subset [0, 2]$, and $g(\lambda) = 1$ for $\lambda \in [0, 1]$, $0 \leq g(\lambda) \leq 1$, $\lambda > 0$. We now let

$$h(\lambda) = g(\lambda) - g(2\lambda) ,$$

which entails $\text{supp}(h) \subset [2^{-1}, 2]$, and use this to define

$$(3.12) \quad F_0(\lambda) = \sqrt{g(\lambda)} , F_j(\lambda) = \sqrt{h(2^{-j}\lambda)} , j \geq 1 ,$$

as well as

$$G_j(\lambda) = [F_j(\lambda)]^2 = F_j^2(\lambda) , j \geq 0 .$$

As a result of the definitions, we get for all $\lambda \geq 0$ the equations

$$(3.13) \quad \sum_{j=0}^n G_j(\lambda) = \sum_{j=0}^n F_j^2(\lambda) = g(2^{-n}\lambda) ,$$

and as a consequence

$$(3.14) \quad \sum_{j \geq 0} G_j(\lambda) = \sum_{j \geq 0} F_j^2(\lambda) = 1 , \quad \lambda \geq 0 ,$$

with finitely many nonzero terms occurring in the sums for each fixed λ . We call the sequence $(G_j)_{j \geq 0}$ a **(dyadic) partition of unity**, and $(F_j)_{j \geq 0}$ a **quadratic (dyadic) partition of unity**. As will become soon apparent, quadratic partitions are useful for the construction of frames.

Using the spectral theorem one has

$$F_j^2(\sqrt{L})f = \mathcal{F}^{-1} (F_j^2(\lambda)\mathcal{F}f(\lambda)) , \quad j \geq 1 ,$$

and thus

$$(3.15) \quad f = \mathcal{F}^{-1}\mathcal{F}f(\lambda) = \mathcal{F}^{-1} \left(\sum_{j \geq 0} F_j^2(\lambda)\mathcal{F}f(\lambda) \right) = \sum_{j \geq 0} F_j^2(\sqrt{L})f$$

Taking inner product with f gives

$$\|F_j(\sqrt{L})f\|^2 = \langle F_j^2(\sqrt{L})f, f \rangle$$

and

$$(3.16) \quad \|f\|^2 = \sum_{j \geq 0} \langle F_j^2(\sqrt{L})f, f \rangle = \sum_{j \geq 0} \|F_j(\sqrt{L})f\|^2 .$$

Similarly, we get the identity

$$\sum_{j \geq 0} G_j(\sqrt{L})f = f .$$

Moreover, since the functions G_j, F_j , have their supports in $[2^{j-1}, 2^{j+1}]$, the elements $F_j(\sqrt{L})f$ and $G_j(\sqrt{L})f$ are bandlimited to $[2^{j-1}, 2^{j+1}]$, whenever $j \geq 1$, and to $[0, 2]$ for $j = 0$.

3.5. Paley-Wiener frames in Hilbert spaces. Using the notation from above and Theorem 3.8, one can describe the following Paley-Wiener frame in an abstract Hilbert space \mathcal{H} .

Theorem 3.9. (*Paley-Wiener nearly Parseval frame in \mathcal{H}*)

For a fixed $\delta \in (0, 1)$ and $j \in \mathbb{N}$ let $\{\phi_k^j\}$ be a set of vectors described in Theorem 3.8 that correspond to $\omega = 2^{j+1}$. Then for functions F_j introduced in (3.12) the family of Paley-Wiener vectors

$$\Phi_k^j = F_j(\sqrt{L})\phi_k^j$$

has the following properties:

- (1) Each vector Φ_k^j belongs to $\mathbf{PW}_{[2^{j-1}, 2^{j+1}]}(\sqrt{L})$, $j \in \mathbb{N}$, $k = 1, \dots$,
- (2) The family $\{\Phi_k^j\}$ is a frame in \mathcal{H} with constants $1 - \delta$ and 1:

$$(3.17) \quad (1 - \delta)\|f\|^2 \leq \sum_{j \geq 0} \sum_k \left| \langle f, \Phi_k^j \rangle \right|^2 \leq \|f\|^2, \quad f \in \mathcal{H}.$$

- (3) The canonical dual frame $\{\Psi_k^j\}$ also consists of bandlimited vectors $\Psi_k^j \in \mathbf{PW}_{[2^{j-1}, 2^{j+1}]}(\sqrt{L})$, $j \in [0, \infty)$, $k = 1, \dots$, and satisfies the inequalities

$$(3.18) \quad \|f\|^2 \leq \sum_{j \geq 0} \sum_k \left| \langle f, \Psi_k^j \rangle \right|^2 \leq (1 - \delta)^{-1} \|f\|^2, \quad f \in \mathcal{H}.$$

- (4) The reconstruction formulas hold for every $f \in \mathcal{H}$

$$(3.19) \quad f = \sum_j \sum_k \langle f, \Phi_k^j \rangle \Psi_k^j = \sum_j \sum_k \langle f, \Psi_k^j \rangle \Phi_k^j.$$

The last item here follows from the general theory of frames [61]. We also note that for reconstruction of a Paley-Wiener vector from a set of samples one can use, besides dual frames, the variational (polyharmonic) splines in Hilbert spaces developed in [101]-[118].

3.6. Interpolation and Approximation spaces. The goal of the section is to establish certain connections between interpolation spaces and approximation spaces to be used later. These connections are well known to specialists and due to Peetre-Sparr [93] and Butzer-Scherer [17]. However, we formulate and prove these relations in a form which is most suitable for our purposes (see [73]). In particular, our formulation is more general than a similar Theorem 9.1 in Ch. 7 in [29]. The result of primary interest for the following is Theorem 3.14, and readers who are not interested in its proof can safely skip this section.

The general theory of interpolation spaces can be found in [7], [16], [72]. The notion of approximation spaces and their relations to interpolations spaces is described in [7], [11], [29], [93].

It is important to realize that the relations between interpolation and approximation spaces cannot be described in the language of normed spaces. We have to make use of quasi-normed linear spaces in order to treat them simultaneously.

A quasi-norm $\|\cdot\|_{\mathbf{E}}$ on linear space \mathbf{E} is a real-valued function on \mathbf{E} such that for any $f, f_1, f_2 \in \mathbf{E}$ the following holds true

- (1) $\|f\|_{\mathbf{E}} \geq 0$;

- (2) $\|f\|_{\mathbf{E}} = 0 \iff f = 0$;
- (3) $\| -f \|_{\mathbf{E}} = \|f\|_{\mathbf{E}}$;
- (4) there exists some $C_{\mathbf{E}} \geq 1$ such that $\|f_1 + f_2\|_{\mathbf{E}} \leq C_{\mathbf{E}}(\|f_1\|_{\mathbf{E}} + \|f_2\|_{\mathbf{E}})$.

For a general quasi-normed linear spaces \mathbf{E} the notation $(\mathbf{E})^\rho$, $\rho > 0$, is used for the space \mathbf{E} endowed with the quasi-norm $\|\cdot\|^\rho$.

Two quasi-normed linear spaces \mathbf{E} and \mathbf{F} form a pair if they are linear subspaces of a common linear space \mathbf{A} and the conditions $\|f_k - g\|_{\mathbf{E}} \rightarrow 0$, and $\|f_k - h\|_{\mathbf{F}} \rightarrow 0$ imply equality $g = h$ (in \mathbf{A}). For any such pair \mathbf{E}, \mathbf{F} one can construct the space $\mathbf{E} \cap \mathbf{F}$ with quasi-norm

$$\|f\|_{\mathbf{E} \cap \mathbf{F}} = \max(\|f\|_{\mathbf{E}}, \|f\|_{\mathbf{F}})$$

and the sum of the spaces, $\mathbf{E} + \mathbf{F}$ consisting of all sums $f_0 + f_1$ with $f_0 \in \mathbf{E}, f_1 \in \mathbf{F}$, and endowed with the quasi-norm

$$\|f\|_{\mathbf{E} + \mathbf{F}} = \inf_{f=f_0+f_1, f_0 \in \mathbf{E}, f_1 \in \mathbf{F}} (\|f_0\|_{\mathbf{E}} + \|f_1\|_{\mathbf{F}}).$$

Quasi-normed spaces \mathbf{H} with $\mathbf{E} \cap \mathbf{F} \subset \mathbf{H} \subset \mathbf{E} + \mathbf{F}$ are called intermediate between \mathbf{E} and \mathbf{F} . If both \mathbf{E} and \mathbf{F} are complete the inclusion mappings are automatically continuous. An additive homomorphism $T : \mathbf{E} \rightarrow \mathbf{F}$ is called bounded if

$$\|T\| = \sup_{f \in \mathbf{E}, f \neq 0} \|Tf\|_{\mathbf{F}} / \|f\|_{\mathbf{E}} < \infty.$$

An intermediate quasi-normed linear space \mathbf{H} interpolates between \mathbf{E} and \mathbf{F} if every bounded homomorphism $T : \mathbf{E} + \mathbf{F} \rightarrow \mathbf{E} + \mathbf{F}$ which is a bounded homomorphism of \mathbf{E} into \mathbf{E} and a bounded homomorphism of \mathbf{F} into \mathbf{F} is also a bounded homomorphism of \mathbf{H} into \mathbf{H} .

On $\mathbf{E} + \mathbf{F}$ one considers the so-called Peetre's K -functional

$$(3.20) \quad K(f, t) = K(f, t, \mathbf{E}, \mathbf{F}) = \inf_{f=f_0+f_1, f_0 \in \mathbf{E}, f_1 \in \mathbf{F}} (\|f_0\|_{\mathbf{E}} + t\|f_1\|_{\mathbf{F}}).$$

The quasi-normed linear space $(\mathbf{E}, \mathbf{F})_{\theta, q}^K$, with parameters $0 < \theta < 1$, $0 < q \leq \infty$, or $0 \leq \theta \leq 1$, $q = \infty$, is introduced as the set of elements f in $\mathbf{E} + \mathbf{F}$ for which

$$(3.21) \quad \|f\|_{\theta, q} = \left(\int_0^\infty (t^{-\theta} K(f, t))^q \frac{dt}{t} \right)^{1/q} < \infty.$$

It turns out that $(\mathbf{E}, \mathbf{F})_{\theta, q}^K$ with the quasi-norm (3.21) interpolates between \mathbf{E} and \mathbf{F} . The following Reiteration Theorem is one of the main results of the theory (see [7], [16], [72], [93]).

Theorem 3.10. *Suppose that $\mathbf{E}_0, \mathbf{E}_1$ are complete intermediate quasi-normed linear spaces for the pair \mathbf{E}, \mathbf{F} . If $\mathbf{E}_i \in \mathcal{K}(\theta_i, \mathbf{E}, \mathbf{F})$ which means*

$$K(f, t, \mathbf{E}, \mathbf{F}) \leq Ct^{\theta_i} \|f\|_{\mathbf{E}_i}, i = 0, 1,$$

where $0 \leq \theta_i \leq 1, \theta_0 \neq \theta_1$, then

$$(\mathbf{E}_0, \mathbf{E}_1)_{h, q}^K \subset (\mathbf{E}, \mathbf{F})_{\theta, q}^K,$$

where $0 < q < \infty, 0 < h < 1, \theta = (1 - h)\theta_0 + h\theta_1$.

If for the same pairs \mathbf{E}, \mathbf{F} and $\mathbf{E}_0, \mathbf{E}_1$ one has $\mathbf{E}_i \in \mathcal{J}(\theta_i, \mathbf{E}, \mathbf{F})$, which means

$$\|f\|_{\mathbf{E}_i} \leq C \|f\|_{\mathbf{E}}^{1-\theta_i} \|f\|_{\mathbf{F}}^{\theta_i}, i = 0, 1,$$

where $0 \leq \theta_i \leq 1, \theta_0 \neq \theta_1$, then for any parameter $0 < q < \infty, 0 < h < 1$

$$(\mathbf{E}, \mathbf{F})_{\theta, q}^K \subset (\mathbf{E}_0, \mathbf{E}_1)_{h, q}^K \quad \text{for } \theta = (1 - h)\theta_0 + h\theta_1.$$

It is important to note that in all cases which will be considered in the present article the space \mathbf{F} will be continuously embedded as a subspace into \mathbf{E} . In this case (3.20) can be introduced by the formula

$$K(f, t) = \inf_{f_1 \in \mathbf{F}} (\|f - f_1\|_{\mathbf{E}} + t\|f_1\|_{\mathbf{F}}),$$

which implies the inequality

$$(3.22) \quad K(f, t) \leq \|f\|_{\mathbf{E}}.$$

This inequality can be used to show that the norm (3.21) is equivalent to the norm

$$(3.23) \quad \|f\|_{\theta, q} = \|f\|_{\mathbf{E}} + \left(\int_0^\varepsilon (t^{-\theta} K(f, t))^q \frac{dt}{t} \right)^{1/q}$$

for any positive ε .

Let us introduce another functional on $\mathbf{E} + \mathbf{F}$, where \mathbf{E} and \mathbf{F} form a pair of quasi-normed linear spaces

$$\mathcal{E}(f, t) = \mathcal{E}(f, t, E, F) = \inf_{g \in \mathbf{F}, \|g\|_{\mathbf{F}} \leq t} \|f - g\|_{\mathbf{E}}.$$

Definition 3.11. *The approximation space $\mathcal{E}_{\alpha, q}(\mathbf{E}, \mathbf{F})$, $0 < \alpha < \infty$, $0 < q \leq \infty$ is the quasi-normed linear spaces of all $f \in \mathbf{E} + \mathbf{F}$ for which the quasi-norm*

$$(3.24) \quad \|f\|_{\mathcal{E}_{\alpha, q}(\mathbf{E}, \mathbf{F})} = \left(\int_0^\infty (t^\alpha \mathcal{E}(f, t))^q \frac{dt}{t} \right)^{1/q}$$

is finite.

The following theorem describes the relations between interpolation and approximation spaces (see [7], Ch. 7).

Theorem 3.12. *For $\theta = 1/(\alpha + 1)$ and $r = \theta q$ one has*

$$(\mathcal{E}_{\alpha, r}(\mathbf{E}, \mathbf{F}))^\theta = (\mathbf{E}, \mathbf{F})_{\theta, q}^K.$$

The following important result is known as the Power Theorem (see [7], Ch. 3).

Theorem 3.13. *Given positive values $\rho_0 > 0$, $\rho_1 > 0$ and interpolation parameters $0 < \theta < 1$ and $0 < q \leq \infty$, the interpolation space obtained from powers of two spaces is the same as the power of an interpolation space: For the parameter values $\rho = (1 - \theta)\rho_0 + \theta\rho_1$, $q' = \rho q$ and $\theta' = \theta\rho_1/\rho$, we have*

$$((\mathbf{E})^{\rho_0}, (\mathbf{F})^{\rho_1})_{\theta, q}^K = ((\mathbf{E}, \mathbf{F})_{\theta', q'}^K)^\rho.$$

The next theorem represents a very abstract version of what is known as an Equivalence Approximation Theorem.

Theorem 3.14. *Suppose that $\mathcal{T} \subset \mathbf{F} \subset \mathbf{E}$ are quasi-normed linear spaces and \mathbf{E} and \mathbf{F} are complete. If there exist $C > 0$ and $\beta > 0$ such that the following Jackson-type inequality is satisfied*

$$(3.25) \quad t^\beta \mathcal{E}(t, f, \mathcal{T}, \mathbf{E}) \leq C\|f\|_{\mathbf{F}}, \quad t > 0, \text{ for all } f \in \mathbf{F},$$

then the following embedding holds true

$$(3.26) \quad (\mathbf{E}, \mathbf{F})_{\theta, q}^K \subset \mathcal{E}_{\theta\beta, q}(\mathbf{E}, \mathcal{T}), \quad 0 < \theta < 1, \quad 0 < q \leq \infty.$$

If there exist $C > 0$ and $\beta > 0$ such that the following Bernstein-type inequality holds

$$(3.27) \quad \|f\|_{\mathbf{F}} \leq C \|f\|_{\mathcal{T}}^{\beta} \|f\|_{\mathbf{E}} \quad \text{for all } f \in \mathcal{T},$$

then the following embedding holds true

$$(3.28) \quad \mathcal{E}_{\theta\beta,q}(\mathbf{E}, \mathcal{T}) \subset (\mathbf{E}, \mathbf{F})_{\theta,q}^K, \quad 0 < \theta < 1, \quad 0 < q \leq \infty.$$

Proof. According to ([7], Ch.7) one has for any $s > 0$

$$(3.29) \quad t = K_{\infty}(f, s) = K_{\infty}(f, s, \mathcal{T}, \mathbf{E}) = \inf_{f=f_1+f_2, f_1 \in \mathcal{T}, f_2 \in \mathbf{E}} \max(\|f_1\|_{\mathcal{T}}, s\|f_2\|_{\mathbf{E}})$$

the following inequality holds

$$(3.30) \quad s^{-1} K_{\infty}(f, s) \leq \lim_{\tau \rightarrow t-0} \inf \mathcal{E}(f, \tau, \mathbf{E}, \mathcal{T}).$$

Since

$$(3.31) \quad K_{\infty}(f, s) \leq K(f, s) \leq 2K_{\infty}(f, s),$$

the Jackson-type inequality (3.25) and the inequality (3.30) imply

$$(3.32) \quad s^{-1} K(f, s, \mathcal{T}, \mathbf{E}) \leq C t^{-\beta} \|f\|_{\mathbf{F}}.$$

The equalities (3.29) and (3.31) imply the estimate

$$(3.33) \quad t^{-\beta} \leq 2^{\beta} (K(f, s, \mathcal{T}, \mathbf{E}))^{-\beta}$$

which along with the previous inequality gives the estimate

$$K^{1+\beta}(f, s, \mathcal{T}, \mathbf{E}) \leq C s \|f\|_{\mathbf{F}}$$

which in turn implies the inequality

$$(3.34) \quad K(f, s, \mathcal{T}, \mathbf{E}) \leq C s^{\frac{1}{1+\beta}} \|f\|_{\mathbf{F}}^{\frac{1}{1+\beta}}.$$

At the same time one has

$$(3.35) \quad K(f, s, \mathcal{T}, \mathbf{E}) = \inf_{f=f_0+f_1, f_0 \in \mathcal{T}, f_1 \in \mathbf{E}} (\|f_0\|_{\mathcal{T}} + s\|f_1\|_{\mathbf{E}}) \leq s\|f\|_{\mathbf{E}},$$

for every $f \in \mathbf{E}$. Inequality (3.34) means that the quasi-normed linear space $(\mathbf{F})^{\frac{1}{1+\beta}}$ belongs to the class $\mathcal{K}(\frac{1}{1+\beta}, \mathcal{T}, \mathbf{E})$ and (3.35) means that the quasi-normed linear space \mathbf{E} belongs to the class $\mathcal{K}(1, \mathcal{T}, \mathbf{E})$. This fact allows to use the Reiteration Theorem to obtain the embedding

$$(3.36) \quad \left((\mathbf{F})^{\frac{1}{1+\beta}}, \mathbf{E} \right)_{\frac{1-\theta}{1+\theta\beta}, q(1+\theta\beta)}^K \subset (\mathcal{T}, \mathbf{E})_{\frac{1}{1+\theta\beta}, q(1+\theta\beta)}^K$$

for every $0 < \theta < 1, 1 < q < \infty$. But the space on the left is the space

$$\left(\mathbf{E}, (\mathbf{F})^{\frac{1}{1+\beta}} \right)_{\frac{\theta(1+\beta)}{1+\theta\beta}, q(1+\theta\beta)}^K,$$

which, according to the Power Theorem, is the space

$$((\mathbf{E}, \mathbf{F})_{\theta,q}^K)^{\frac{1}{1+\theta\beta}}.$$

All these results along with the equivalence of interpolation and approximation spaces give the embedding

$$(\mathbf{E}, \mathbf{F})_{\theta,q}^K \subset \left((\mathcal{T}, \mathbf{E})_{\frac{1}{1+\theta\beta}, q(1+\theta\beta)}^K \right)^{1+\theta\beta} = \mathcal{E}_{\theta\beta,q}(\mathbf{E}, \mathcal{T}),$$

which proves the embedding (3.26).

Conversely, the Bernstein-type inequality (3.27) implies for $\gamma = \frac{1}{1+\beta}$ the inequality

$$(3.37) \quad \|f\|_{\mathbf{F}}^\gamma \leq C \|f\|_{\mathcal{T}}^\gamma \|f\|_{\mathbf{E}}^\gamma.$$

Along with the obvious equality $\|f\|_{\mathbf{E}} = \|f\|_{\mathcal{T}}^0 \|f\|_{\mathbf{E}}$ and the Iteration Theorem one obtains the embedding

$$(\mathcal{T}, \mathbf{E})_{\frac{1}{1+\theta\beta}, q(1+\theta\beta)}^K \subset \left((\mathbf{F})^{\frac{1}{1+\beta}}, \mathbf{E} \right)_{\frac{1-\theta}{1+\theta\beta}, q(1+\theta\beta)}^K.$$

In order to derive the embedding (3.28), one can then use the same arguments as above. This completes the proof. \square

3.7. Besov subspaces in Hilbert spaces. We introduce the inhomogeneous Besov space $\mathcal{B}_{\mathcal{H},q}^\alpha(\sqrt{L})$ as an interpolation space between the Hilbert space \mathcal{H} and "Sobolev space" H^r , defined as the domain of the operator $(I+L)^{r/2}$ endowed with the graph norm, where r can be any natural number such that $0 < \alpha < r, 1 \leq q \leq \infty$. More precisely, we have [16], [147],

$$\mathcal{B}_{\mathcal{H},q}^\alpha(\sqrt{L}) = (\mathcal{H}, H^r)_{\theta,q}^K, \quad 0 < \theta = \alpha/r < 1, \quad 1 \leq q \leq \infty.$$

We also introduce a notion of best approximation

$$(3.38) \quad \mathcal{E}(f, \omega) = \inf_{g \in \mathbf{PW}_\omega(\sqrt{L})} \|f - g\|_{\mathcal{H}}.$$

Our goal is to apply Theorem 3.14 in the situation where $E = \mathcal{H}$, $F = H^r$ and $\mathcal{T} = \mathbf{PW}_\omega(\sqrt{L})$ is a natural abelian group as the additive group of a vector space, with the quasi-norm

$$\|f\|_{\mathcal{T}} = \inf \left\{ \omega' > 0 : f \in \mathbf{PW}_{\omega'}(\sqrt{L}) \right\}.$$

To be more precise it is the space of finite sequences of Fourier coefficients $\mathbf{c} = (c_1, \dots, c_m) \in \mathbf{PW}_\omega(\sqrt{L})$ where m is the greatest index such that the eigenvalue $\lambda_m \leq \omega$. The quasi-norm $\|\mathbf{c}\|_{\mathbf{PW}_\omega(\sqrt{L})}$ where $\mathbf{c} = (c_1, \dots, c_m) \in \mathbf{PW}_\omega(\sqrt{L})$ is defined as square root from the highest eigenvalue λ_j for which the corresponding Fourier coefficient $c_j \neq 0$ but $c_{j+1} = \dots = c_m = 0$:

$$\|\mathbf{c}\|_{E_\omega(L)} = \|(c_1, \dots, c_m)\|_{E_\omega(L)} = \max \left\{ \sqrt{\lambda_j} : c_j \neq 0, \quad c_{j+1} = \dots = c_m = 0 \right\}.$$

Remark 3.15. *Let us emphasize that the reason we need the language of quasi-normed spaces is because $\|\cdot\|_{\mathcal{T}}$ is clearly not a norm, only a quasi-norm on $\mathbf{PW}_\omega(\sqrt{L})$.*

The Plancherel Theorem allows us to verify a generalization of the Bernstein inequality for bandlimited functions in $f \in \mathbf{PW}_\omega(\sqrt{L})$.

Lemma 3.16. ([97], [104]) *A vector f belongs to the space $\mathbf{PW}_\omega(\sqrt{L})$ if and only if the following Bernstein inequality holds*

$$\|L^{r/2} f\|_{\mathcal{H}} \leq \omega^r \|f\|_{\mathcal{H}}, \quad r \in \mathbb{R}_+.$$

Proof. Assume that vector f belongs to the space $\mathbf{PW}_\omega(\sqrt{L})$ and $\mathcal{F}f = x \in X$. Then

$$\left(\int_0^\infty \lambda^{2r} \|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda) \right)^{1/2} = \left(\int_0^\omega \lambda^{2r} \|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda) \right)^{1/2} \leq \omega^r \|x\|_X, \quad r \in \mathbb{R}_+,$$

which gives Bernstein inequality for f .

Conversely, if f satisfies Bernstein inequality then $x = \mathcal{F}f$ satisfies $\|x\|_{X_k} \leq \omega^k \|x\|_X$. Suppose that there exists a set $\sigma \subset [0, \infty] \setminus [0, \omega]$ whose m -measure is not zero and $x|_\sigma \neq 0$. We can assume that $\sigma \subset [\omega + \epsilon, \infty)$ for some $\epsilon > 0$. Then for any $r \in \mathbb{R}_+$ we have

$$\int_\sigma \|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda) \leq \int_{\omega+\epsilon}^\infty \lambda^{-2r} \|\lambda^r x(\lambda)\|_{X(\lambda)}^2 d\mu \leq \|x\|_X^2 (\omega/\omega + \epsilon)^{2r}$$

which shows that or $x(\lambda)$ is zero on σ or σ has measure zero. \square

One also has an analogue of the Jackson inequality

$$\mathcal{E}(f, \omega) \leq \omega^{-r} \|f\|_{H^r}, \quad f \in H^r.$$

Indeed, as in [97] one has for $f \in H^r$

$$\begin{aligned} \mathcal{E}(f, \omega) &\leq \left(\int_\omega^\infty \|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda) \right)^{1/2} = \\ &\left(\int_\omega^\infty \lambda^{-2r} \lambda^{2r} \|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda) \right)^{-r} \leq \omega^{-r} \|L^{r/2} f\|_{\mathcal{H}}, \quad r \in \mathbb{R}_+. \end{aligned}$$

These two inequalities and Theorem 3.14 imply the following result (compare to [96], [118], [122]).

Theorem 3.17. *For $\alpha > 0, 1 \leq q \leq \infty$ the norm of $\mathcal{B}_{\mathcal{H},q}^\alpha(\sqrt{L})$, is equivalent to*

$$(3.39) \quad \|f\|_{\mathcal{H}} + \left(\sum_{j=0}^\infty (2^{j\alpha} \mathcal{E}(f, 2^j))^q \right)^{1/q}.$$

Let the functions F_j be as in Subsection 3.4. Note, that

$$(3.40) \quad F_j(\sqrt{L}) : \mathcal{H} \rightarrow \mathbf{PW}_{[2^{j-1}, 2^{j+1}]}(\sqrt{L}), \quad \|F_j(\sqrt{L})\| \leq 1,$$

and

$$\sum_{j \geq 0} F_j^2(\lambda) = 1, \quad \lambda \geq 0.$$

Theorem 3.18. *For $\alpha > 0, 1 \leq q \leq \infty$ the norm of $\mathcal{B}_{\mathcal{H},q}^\alpha(\sqrt{L})$, is equivalent to*

$$(3.41) \quad f \mapsto \left(\sum_{j=0}^\infty \left(2^{j\alpha} \|F_j(\sqrt{L})f\|_{\mathcal{H}} \right)^q \right)^{1/q},$$

with the standard modifications for $q = \infty$.

Proof. Recall from (3.15) that

$$\|f\|^2 = \sum_{j \geq 0} \|F_j(\sqrt{L})f\|^2.$$

We obviously have

$$\mathcal{E}(f, 2^l) \leq \sum_{j > l} \|F_j(\sqrt{L})f\|_{\mathcal{H}}.$$

By using a discrete version of Hardy's inequality [16] we obtain the estimate

$$(3.42) \quad \|f\| + \left(\sum_{l=0}^{\infty} (2^{l\alpha} \mathcal{E}(f, 2^l))^q \right)^{1/q} \leq C \left(\sum_{j=0}^{\infty} (2^{j\alpha} \|F_j(\sqrt{L})f\|_{\mathcal{H}})^q \right)^{1/q}.$$

Conversely, for any $g \in \mathbf{PW}_{2^{j-1}}(\sqrt{L})$ we have

$$\|F_j(\sqrt{L})f\|_{\mathcal{H}} = \|F_j(\sqrt{L})(f - g)\|_{\mathcal{H}} \leq \|f - g\|_{\mathcal{H}}.$$

This implies the estimate

$$\|F_j(\sqrt{L})f\|_{\mathcal{H}} \leq \mathcal{E}(f, 2^{j-1}),$$

which shows that the inequality opposite to (3.42) holds. This completes the proof. \square

Theorem 3.19. *For $\alpha > 0, 1 \leq q \leq \infty$ the norm of $\mathcal{B}_{\mathcal{H},q}^{\alpha}(\sqrt{L})$ is equivalent to*

$$(3.43) \quad \left(\sum_{j=0}^{\infty} 2^{j\alpha q} \left(\sum_k \left| \langle f, \Phi_k^j \rangle \right|^2 \right)^{q/2} \right)^{1/q} \asymp \|f\|_{B_q^{\alpha}},$$

with the standard modifications for $q = \infty$.

Proof. For $f \in \mathcal{H}$ and operator $F_j(\sqrt{L})$ we apply (3.10) to $F_j(\sqrt{L})f \in \mathbf{PW}_{2^{j+1}}(\sqrt{L})$ to obtain

$$(3.44) \quad (1 - \delta) \|F_j(\sqrt{L})f\|_{\mathcal{H}}^2 \leq \sum_k \left| \langle F_j(\sqrt{L})f, \phi_k^j \rangle \right|^2 \leq \|F_j(\sqrt{L})f\|_{\mathcal{H}}^2.$$

Since $\Phi_k^j = F_j(\sqrt{L})\phi_k^j$ we obtain the following inequality

$$\sum_k \left| \langle f, \Phi_k^j \rangle \right|^2 \leq \|F_j(\sqrt{L})f\|_{\mathcal{H}}^2 \leq \frac{1}{1 - \delta} \sum_k \left| \langle f, \Phi_k^j \rangle \right|^2 \quad \text{for all } f \in \mathcal{H}.$$

Our statement follows now from Theorem 3.18. \square

4. MANIFOLDS, FUNCTION SPACES AND OPERATORS

The goal of this section is to introduce some "real life" situations in which we develop space-frequency analysis.

4.1. Riemannian manifolds without boundary. Let \mathbf{M} , $\dim \mathbf{M} = n$, be a connected C^∞ -smooth Riemannian manifold with a $(2,0)$ metric tensor g that defines an inner product on every tangent space $T_x(\mathbf{M})$, $x \in \mathbf{M}$. The corresponding Riemannian distance d on \mathbf{M} and the Riemannian measure $d\mu$ on \mathbf{M} are given by

$$(4.1) \quad d(x, y) = \inf \int_a^b \sqrt{g\left(\frac{d\alpha}{dt}, \frac{d\alpha}{dt}\right)} dt, \quad d\mu = \sqrt{\det(g_{ij})} dx,$$

where the infimum is taken over all C^1 -curves $\alpha : [a, b] \rightarrow \mathbf{M}$, $\alpha(a) = x$, $\alpha(b) = y$, the $\{g_{ij}\}$ are the components of the tensor g in a local coordinate system and dx is the Lebesgue measure on \mathbb{R}^d . Let $\exp_x : T_x(\mathbf{M}) \rightarrow \mathbf{M}$, be the exponential geodesic map i. e. $\exp_x(u) = \gamma(1)$, $u \in T_x(\mathbf{M})$, where $\gamma(t)$ is the geodesic starting at x with the initial vector $u : \gamma(0) = x$, $\frac{d\gamma(0)}{dt} = u$. We denote by inj the largest real number r such that \exp_x is a diffeomorphism of a suitable open neighborhood of 0 in $T_x\mathbf{M}$ onto $B(x, \rho)$, for all $\rho < r$ and $x \in \mathbf{M}$. Thus for every choice of an orthonormal basis (with respect to the inner product defined by g) of $T_x(\mathbf{M})$ the exponential map \exp defines a coordinate system on $B(x, \rho)$ which is called *geodesic*. The volume of the ball $B(x, \rho)$ will be denoted by $|B(x, \rho)|$. Throughout the paper we will consider only geodesic coordinate systems.

We will consider only Riemannian manifolds of bounded geometry. Let us recall that a manifold has bounded geometry if

- (a) \mathbf{M} is complete and connected;
- (b) the injectivity radius $\text{inj}(\mathbf{M})$ is positive;
- (c) for any $\rho \leq \text{inj}(\mathbf{M})$, any $k \geq 0$ there exists a constant $C(\rho, k)$ such that

$$\sup_{z \in \vartheta_y^{-1}(B(x, \rho) \cap B(y, \rho))} \sup_{|\alpha| \leq k} |\partial^\alpha (\vartheta_x^{-1} \vartheta_y)(z)| \leq C(\rho, k),$$

for all geodesic coordinate systems $\vartheta_x : T_x(\mathbf{M}) \rightarrow B(x, \rho)$, $\vartheta_y : T_y(\mathbf{M}) \rightarrow B(y, \rho)$.

Examples of manifolds of bounded geometry are: compact Riemannian manifolds, all Lie groups with left (right) invariant Riemannian structure and their homogeneous manifolds, covering spaces of all compact manifolds, bounded domains in \mathbb{R}^n with smooth boundaries.

We will also need the following condition:

- (d) The Riemannian measure fulfills the local doubling property i.e. there exists a constant $C(\mathbf{M})$ such that for any sufficiently small $\rho < \text{inj}(\mathbf{M})$ and any $0 < \sigma < \lambda < \rho$ the following inequality (local doubling property) holds true

$$(4.2) \quad |B(x, \lambda)| \leq C(\mathbf{M}) (\lambda/\sigma)^n |B(x, \sigma)|, \quad n = \dim \mathbf{M}.$$

Note that the Bishop-Gromov Comparison Theorem implies (see [64]) that this condition is satisfied whenever the Ricci curvature Ric is bounded from below. More precisely, if

$$(4.3) \quad \text{Ric} \geq -kg, \quad k \geq 0$$

the local doubling property (d) is satisfied for $0 < \sigma < \lambda < \delta < \text{inj}(\mathbf{M})$:

$$(4.4) \quad |B(x, \rho)| \leq (\rho/\sigma)^n e^{(k\delta(n-1))^{1/2}} |B(x, \sigma)|, \quad n = \dim \mathbf{M}.$$

Lemma 4.1. ([108], [125]) *If \mathbf{M} has bounded geometry and condition (4.3) holds, then there exists a natural number $N_{\mathbf{M}}$ such that for any $0 < r < \text{inj}(\mathbf{M})$ there exists a set of points $M_r = \{x_i\}$ with the following properties*

- (1) *the balls $B(x_i, r/4)$ are disjoint,*
- (2) *the balls $B(x_i, r/2)$ form a cover of \mathbf{M} ,*
- (3) *the height of the cover by the balls $B(x_i, r)$ is at most $N_{\mathbf{M}}$.*

The important feature of this Lemma is the claim that height $N_{\mathbf{M}}$ is independent on r for $0 < r < \text{inj}(\mathbf{M})$.

Definition 4.2. *Any set of points $\{x_i\} \in \mathbf{M}$ which satisfies the above properties (1)-(3) will be denoted as \mathbf{M}_r and called $(r, N_{\mathbf{M}})$ -lattice of \mathbf{M} .*

To construct Sobolev spaces $W_p^k(\mathbf{M})$, $k \in \mathbb{N}$, we fix a ρ -lattice $\mathbf{M}_\rho = \{y_\nu\}$, $0 < \rho < \text{inj}(\mathbf{M})$ and introduce a partition of unity φ_ν that is subordinate to the family $\{B(y_\nu, \rho/2)\}$ and has the following properties:

- (1) $\varphi_\nu \in C_0^\infty(B(y_\nu, \rho/2))$,
- (2) $\sup_x \sup_{|\alpha| \leq k} |\varphi_\nu^{(\alpha)}(x)| \leq C(k)$, in geodesic coordinates, where $C(k)$ is independent of ν for every k .

Such a partition is called *bounded uniform partition of unity*, for short a BUPU, or more precisely a $(\rho, N_{\mathbf{M}})$ -BUPU. We introduce the Sobolev space $W_p^k(\mathbf{M})$, $k \in \mathbb{N}$, $1 \leq p \leq \infty$ as the set of all $f \in L_p(\mathbf{M})$ for which

$$(4.5) \quad \|f\|_{W_p^k(\mathbf{M})} = \left(\sum_{\nu} \|\varphi_\nu f\|_{W_p^k(\mathbb{R}^n)}^p \right)^{1/p} < \infty.$$

Here the norm $\|\varphi_\nu f\|_{W_p^k(\mathbb{R}^n)}^p$ is to be understood as the Sobolev norm of a pullback of $\varphi_\nu f$ to \mathbb{R}^n via a geodesic coordinate system on $B(y_\nu, \rho/2)$. A way to introduce Besov spaces $\mathcal{B}_{p,q}^\alpha(\mathbf{M})$ is by using Peetre's K -functional

$$(4.6) \quad \mathcal{B}_{p,q}^\alpha(\mathbf{M}) = (L_p(\mathbf{M}), W_p^r(\mathbf{M}))_{\alpha/r, q}^K, \quad 0 < \alpha < r \in \mathbb{N}, \quad 1 \leq p \leq \infty, \quad 0 < q \leq \infty.$$

The fact that one can use in (4.6) any natural $r > \alpha$ is well known [16], [147].

Note that [145]-[147] gives an equivalent definition for \mathbf{M} compact:

$$(4.7) \quad \|f\|_{\mathcal{B}_{p,q}^\alpha(\mathbf{M})} = \left(\sum_{\nu} \|\varphi_\nu f\|_{\mathcal{B}_{p,q}^\alpha(\mathbb{R}^n)}^p \right)^{1/p} < \infty.$$

However, due to the lack of the Localization Principle (see [147]) the later definition cannot be used in the case of non-compact manifolds.

We will explore second-order differential elliptic operators which are self-adjoint and non-negative definite in the corresponding space $L_2(\mathbf{M})$. The best known example of such an operator is the Laplace-Beltrami which is given in a local coordinate system by the formula

$$Lf = - \sum_{m,k} \frac{1}{\sqrt{\det(g_{ij})}} \partial_m \left(\sqrt{\det(g_{ij})} g^{mk} \partial_k f \right)$$

where the g_{ij} denote the components of the metric tensor, $\det(g_{ij})$ is the determinant of the matrix (g_{ij}) , and g^{mk} denote the components of the matrix inverse to (g_{ij}) . The Laplace-Beltrami is a self-adjoint positive definite operator in the

corresponding space $L_2(\mathbf{M})$ constructed from g . The domains of the powers $L^{s/2}$ coincide with the Sobolev spaces $H^s(\mathbf{M}) = W_2^s(\mathbf{M})$, for $s \in \mathbb{R}$.

4.2. Compact Riemannian manifolds. In this section we consider only compact Riemannian manifolds. Let now L be a smooth, self-adjoint, non-negative, second order elliptic differential operator on the space $L_2(\mathbf{M})$, over a compact Riemannian manifold. The spectrum of the positive square root \sqrt{L} operator is given by a sequence

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

approaching infinity. Let u_0, u_1, u_2, \dots be a corresponding complete system of real-valued orthonormal eigenfunctions, and let $\mathbf{E}_\omega(\sqrt{L})$, $\omega > 0$, be the span of all eigenfunctions of \sqrt{L} whose corresponding eigenvalues are not greater than ω . Clearly $\mathbf{E}_\omega = \mathbf{P}\mathbf{W}_\omega(\sqrt{L})$, and these subspaces are finite-dimensional, and contained in $C^\infty(\mathbf{M}) \subset L_p(\mathbf{M})$, $1 \leq p \leq \infty$. Since the operator L is of order two, the dimension \mathcal{N}_ω of the space $\mathbf{E}_\omega(\sqrt{L})$ is given asymptotically by Weyl's formula [141], which says, in sharp form: for some $c > 0$, and $n = \dim \mathbf{M}$ one has

$$(4.8) \quad \mathcal{N}_\omega(L) = c\omega^n + O(\omega^{(n-1)}).$$

Since $\mathcal{N}_{\lambda_l} = l + 1$, we conclude that, for some constants $c_1, c_2 > 0$,

$$(4.9) \quad c_1 l^{1/n} \leq \lambda_l \leq c_2 l^{1/n}, \quad l \geq 0.$$

Since $L^m u_l = \lambda_l^{2m} u_l$, and L^m is an elliptic differential operator of degree $2m$, Sobolev's lemma, combined with the last fact, implies that for any integer $k \geq 0$, there exist C_k , and $\nu_k > 0$ such that

$$(4.10) \quad \|u_l\|_{C^k(\mathbf{M})} \leq C_k(l+1)^{\nu_k}.$$

4.3. Compact homogeneous manifolds. The most complete results will be obtained for compact homogeneous manifolds.

A *compact homogeneous manifold* \mathbf{M} is a C^∞ -compact manifold on which a compact Lie group G , $\dim G = d$, acts transitively. In this case \mathbf{M} is necessary of the form G/K , where K is a closed subgroup of G . The notation $L_2(\mathbf{M})$ is used for the usual Hilbert spaces, with invariant measure dx on \mathbf{M} .

The Lie algebra \mathfrak{g} of a compact Lie group G is then a direct sum $\mathfrak{g} = \mathfrak{a} + [\mathfrak{g}, \mathfrak{g}]$, where \mathfrak{a} is the center of \mathfrak{g} , and $[\mathfrak{g}, \mathfrak{g}]$ is a semi-simple algebra. Let Q be a positive-definite quadratic form on \mathfrak{g} which, on $[\mathfrak{g}, \mathfrak{g}]$, is opposite to the Killing form. Let X_1, \dots, X_d be a basis of \mathfrak{g} , which is orthonormal with respect to Q . Since the form Q is $Ad(G)$ -invariant, the operator

$$(4.11) \quad -X_1^2 - X_2^2 - \dots - X_d^2, \quad d = \dim G$$

is a bi-invariant operator on G , which is known as the *Casimir operator*. This implies in particular that the corresponding operator on $L_2(\mathbf{M})$,

$$(4.12) \quad \mathcal{L} = -D_1^2 - D_2^2 - \dots - D_d^2, \quad D_j = D_{X_j}, \quad d = \dim G,$$

commutes with all operators $D_j = D_{X_j}$. The operator \mathcal{L} , which is usually called the *Laplace operator*, is the image of the Casimir operator under the differential of the quasi-regular representation in $L_2(\mathbf{M})$. It is important to realize that in general, the operator \mathcal{L} is not necessarily the Laplace-Beltrami operator of the natural invariant metric on \mathbf{M} . But it coincides with this operator at least in the following cases: 1) If \mathbf{M} is a d -dimensional torus, 2) If the manifold \mathbf{M} is itself a compact semi-simple

Lie group G ([65], Chap. 3). If $\mathbf{M} = G/K$ is a compact symmetric space of rank one ([65]).

4.4. An example: the sphere \mathbb{S}^d . We will specify the general setup in the case of standard unit sphere. Let us write

$$\mathbb{S}^d = \{x \in \mathbb{R}^{d+1} : \|x\| = 1\}.$$

We denote the space of spherical harmonics of degree l by the symbol \mathcal{P}_l . They are the restrictions of harmonic homogeneous polynomials of degree l in \mathbb{R}^d to \mathbb{S}^d . The Laplace-Beltrami operator $\Delta_{\mathbb{S}}$ on \mathbb{S}^d is the pullback of the regular Laplace operator Δ in \mathbb{R}^d , given by

$$\Delta_{\mathbb{S}} f(x) = \Delta \tilde{f}(x), \quad x \in \mathbb{S}^d,$$

where $\tilde{f}(x)$ is the homogeneous extension of f : $\tilde{f}(x) = f(x/\|x\|)$. Another way to compute $\Delta_{\mathbb{S}} f(x)$ is to express both $\Delta_{\mathbb{S}}$ and f in a spherical coordinate system.

Each \mathcal{P}_l is the eigenspace of $\Delta_{\mathbb{S}}$ that corresponds to the eigenvalue $-l(l+d-1)$. This space has dimension n_d , given by

$$n_d(l) = (d+2l-1) \frac{(d+l-2)!}{l!(d-1)!}.$$

An orthonormal basis for the eigenspace \mathcal{P}_l , $l = 0, 1, 2, \dots$, will be denote by $\mathcal{Y}_{n,l}$, $n = 1, \dots, n_d(l)$.

Let e_1, \dots, e_{d+1} be the standard orthonormal basis in \mathbb{R}^{d+1} . Writing $SO(d+1)$ and $SO(d)$ for the groups of rotations of \mathbb{R}^{d+1} and \mathbb{R}^d respectively we have $\mathbb{S}^d = SO(d+1)/SO(d)$. On \mathbb{S}^d we consider the vector fields

$$X_{i,j} = x_j \partial_{x_i} - x_i \partial_{x_j}, \quad i < j,$$

which are generators of one-parameter groups of rotations $\exp tX_{i,j} \in SO(d+1)$ in the plane (x_i, x_j) . These groups are defined by the formulas

$$\exp \tau X_{i,j} \cdot (x_1, \dots, x_{d+1}) = (x_1, \dots, x_i \cos \tau - x_j \sin \tau, \dots, x_i \sin \tau + x_j \cos \tau, \dots, x_{d+1}).$$

Let $e^{\tau X_{i,j}}$ be a one-parameter group which is a representation of $\exp \tau X_{i,j}$ in a space $L_p(\mathbb{S}^d)$. It acts on $f \in L_p(\mathbb{S}^d)$ by the following formula

$$e^{\tau X_{i,j}} f(x_1, \dots, x_{d+1}) = f(x_1, \dots, x_i \cos \tau - x_j \sin \tau, \dots, x_i \sin \tau + x_j \cos \tau, \dots, x_{d+1}).$$

The Laplace-Beltrami operator $\Delta_{\mathbb{S}}$ can be identified with an operator in $L_p(\mathbb{S}^d)$, given by the formula

$$\Delta_{\mathbb{S}} = L = - \sum_{(i,j)} X_{i,j}^2.$$

Note, that $L\mathcal{Y}_{n,l} = -l(l+d-1)\mathcal{Y}_{n,l}$. Since the vector fields $X_{i,j}$ generate tangent space at every point of \mathbb{S}^d the operator L is elliptic and domains of its natural powers coincide with the regular Sobolev spaces $W_p^k(\mathbb{S}^d)$. Clearly, the norm of $B_{2,2}^\alpha(\mathbb{S}^d)$ is equivalent to

$$(4.13) \quad \left(\sum_{l=0}^{\infty} \sum_{n=1}^{n_d(l)} (l+1)^{2\alpha} |c_{n,l}(f)|^2 \right)^{1/2}, \quad c_{n,l}(f) = \int_{\mathbb{S}^d} f \cdot \mathcal{Y}_{n,l} ds, \quad f \in L_2(\mathbb{S}^d).$$

Another description of the Besov spaces $B_{p,q}^\alpha(\mathbb{S}^d)$ can be given using the modulus of continuity associated to the one-parameter groups $e^{\tau X_{i,j}}$ (see Section 10.2).

4.5. Bounded domains with smooth boundaries. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ , assumed to be a smooth $(d-1)$ -dimensional oriented manifold. Let $\overline{\Omega} = \Omega \cup \Gamma$ and $L_2(\Omega)$ be the space of functions square-integrable with respect to Lebesgue measure $dx = dx_1 \dots dx_n$. If k is a natural number the notation $H^k(\Omega)$ will be used for the Sobolev space of distributions on Ω (see [76] for a precise definition) with the norm

$$\|f\|_{H^k(\Omega)} = \left(\|f\|^2 + \sum_{1 \leq |\alpha| \leq k} \|\partial^\alpha f\|^2 \right)^{1/2}$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a natural vector and ∂^α is a mixed partial derivative

$$\left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n}.$$

Under our assumptions the space $C_0^\infty(\overline{\Omega})$ of infinitely smooth functions with support in $\overline{\Omega}$ is dense in $H^k(\Omega)$. The closure of the space $C_0^\infty(\Omega)$ of smooth functions with support in Ω in $H^k(\Omega)$ is denoted by $H_0^k(\Omega)$.

Since Γ can be treated as a smooth Riemannian manifold one can introduce a Sobolev scale of spaces $H^s(\Gamma)$, $s \in \mathbb{R}$, as, for example, the domains of the Laplace-Beltrami operator \mathcal{L} of a Riemannian metric on Γ .

The trace theorem provides a continuous and surjective trace operator

$$\gamma : H^s(\Omega) \rightarrow H^{s-1/2}(\Gamma), \quad s > 1/2,$$

such that for all functions $f \in H^s(\Omega)$ which are smooth up to the boundary the value $\gamma(f)$ is simply a restriction of f to Γ .

One considers a strictly elliptic self-adjoint positive definite operator L generated by an expression

$$(4.14) \quad Lf = - \sum_{k,i=1}^d \partial_{x_k} (a_{k,i}(x) \partial_{x_i} f),$$

with coefficients in $C^\infty(\Omega)$ where the matrix $(a_{j,k}(x))$ is real, symmetric and positive definite on $\overline{\Omega}$. The operator L is defined as the Friedrichs extension of L , initially defined on $C_0^\infty(\Omega)$, to the set of all functions f in $H^2(\Omega)$ with constraint $\gamma f = 0$. The Green formula implies that this operator is self-adjoint. The domain of its positive square root \sqrt{L} is the set of all functions f in $H^1(\Omega)$ for which $\gamma f = 0$.

Thus, one obtains a self-adjoint positive definite operator in the Hilbert space $L_2(\Omega)$ with a discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \dots$, with $\lim_{n \rightarrow \infty} \lambda_n = +\infty$.

An important example of such a situation is the Dirichlet Laplacian on the unit ball in \mathbb{R}^n . In spherical coordinates (r, ϑ) , $\vartheta \in \mathbb{S}^{n-1}$, one has

$$(4.15) \quad Lf = \partial_r^2 f + \frac{n-1}{r} \partial_r f - \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}} f,$$

with the boundary condition

$$(4.16) \quad f|_{r=1} = 0,$$

where $\Delta_{\mathbb{S}^{n-1}}$ is the Laplace-Beltrami operator on the unit sphere \mathbb{S}^{n-1} in \mathbb{R}^n .

It is known that the eigenvalues of such an operator are given by the formula

$$\lambda = j_{m+\frac{n-2}{2}, l}^2,$$

where $j_{\nu, l}$ is the l -th positive zero of the Bessel function of first kind J_ν of order ν and the corresponding eigenfunctions are of the form

$$(4.17) \quad u_{m, k, l} = c_{m, k, l} r^{-\frac{n-2}{2}} J_{m+\frac{n-2}{2}} \left(j_{m+\frac{n-2}{2}, l} r \right) Y_{m, n}^k(\vartheta),$$

with $m = 0, 1, \dots$, $1 \leq k \leq k_{m, n}$, $l = 1, 2, \dots$. The constants $c_{m, k, l}$ are chosen to normalize the functions $u_{m, k, l}$ with respect to $\|\cdot\|_2$.

4.6. The Poincaré hyperbolic upper half-plane. To illustrate our results about sampling and frames on non-compact symmetric spaces we will use the hyperbolic plane in its upper half-plane realization.

Let $G = SL(2, \mathbb{R})$ be the special linear group of all 2×2 real matrices with determinant 1, and let $K = SO(2)$ denote the group of all rotations of \mathbb{R}^2 . The factor $\mathbb{H} = G/K$ is known as the 2-dimensional hyperbolic space and can be described in many different ways. In the present paper we consider the realization of \mathbb{H} which is called Poincaré upper half-plane (see [65], [144]).

As a Riemannian manifold \mathbb{H} is identified with the regular upper half-plane of the complex plane

$$\mathbb{H} = \{x + iy | x, y \in \mathbb{R}, y > 0\}$$

with a new Riemannian metric

$$ds^2 = y^{-2}(dx^2 + dy^2)$$

and corresponding Riemannian measure

$$d\mu = y^{-2} dx dy.$$

If we define the action of $\sigma \in G$ on $z \in \mathbb{H}$ as a fractional linear transformation

$$\sigma \cdot z = (az + b)/(cz + d),$$

then the metric ds^2 and the measure $d\mu$ are invariant under the action of G on \mathbb{H} . The point $i = \sqrt{-1} \in \mathbb{H}$ is invariant for all $\sigma \in K$. The Haar measure dg on G can be normalized in a way that the following important formula holds true

$$\int_{\mathbb{H}} f(z) y^{-2} dx dy = \int_G f(g \cdot i) dg.$$

In the corresponding Hilbert space $L_2(\mathbb{H})$ with the inner product

$$\langle f, h \rangle = \int_{\mathbb{H}} f(x, y) \overline{h(x, y)} y^{-2} dx dy$$

we consider the Laplace-Beltrami operator

$$\Delta = -y^2 (\partial_x^2 + \partial_y^2)$$

of the metric ds^2 . The operator Δ , acting on $L_2(\mathbb{H}) = L_2(\mathbb{H}, d\mu)$ is initially defined on $C_0^\infty(\mathbb{H})$ and has a self-adjoint closure in $L_2(\mathbb{H})$.

The Helgason-Fourier transform of f for $\lambda \in \mathbb{C}$, $\varphi \in (0, 2\pi]$, is defined by the formula

$$\hat{f}(\lambda, \varphi) = \int_{\mathbb{H}} f(z) \overline{Im(k_\varphi z)^\lambda} y^{-2} dx dy,$$

where $k_\varphi \in SO(2)$ is the rotation of \mathbb{R}^2 by angle φ . We have the following inversion formula for all $f \in C_0^\infty(\mathbb{H})$

$$f(z) = (8\pi^2)^{-1} \int_{\lambda \in \mathbb{R}} \int_0^{2\pi} \hat{f}(i\lambda + 1/2, \varphi) Im(k_\varphi z)^{i\lambda+1/2} \lambda \tanh \pi \lambda d\varphi d\lambda.$$

The Plancherel Theorem states that the map $f \rightarrow \hat{f}$ can be extended to an isometry of $L_2(\mathbb{H})$ with respect to the invariant measure $d\mu$ onto $L_2(\mathbb{R} \times (0, 2\pi])$ with respect to the measure

$$\frac{1}{8\pi^2} \lambda \tanh \pi \lambda d\lambda d\varphi.$$

If f is a function on \mathbb{H} and φ is a $K = SO(2)$ -invariant function on \mathbb{H} , their convolution is defined by the formula

$$f * \varphi(g \cdot i) = \int_{SL(2, \mathbb{R})} f(gu^{-1} \cdot i) \varphi(u) du, \quad i = \sqrt{-1},$$

where du is the Haar measure on $SL(2, \mathbb{R})$. For the Helgason-Fourier transform one has:

$$\widehat{f * \varphi} = \hat{f} \cdot \hat{\varphi}.$$

The following formula holds true

$$(4.18) \quad \widehat{\Delta f} = \left(\lambda^2 + \frac{1}{4} \right) \hat{f}.$$

The one parameter group of operators $e^{it\Delta}$ acts on functions via the formula

$$e^{it\Delta} f(z) = f * G_t,$$

where

$$G_t(ke^{-r}i) = (4\pi)^{-1} \int_{s \in \mathbb{R}} e^{-i(s^2+1/4)t} P_{is-1/2}(\cosh r) s \tanh \pi s ds,$$

here $k \in SO(2)$, r is the geodesic distance, $ke^{-r}i$ is representation of points of \mathbb{H} in the geodesic polar coordinate system on \mathbb{H} , and $P_{is-1/2}$ is the associated Legendre function. In these terms our Theorem 3.4 takes the following form.

Theorem 4.3. *For $f \in L_2(\mathbb{H}, d\mu)$ the following conditions are equivalent.*

- (1) *f belongs to the space $PW_\omega(\Delta)$.*
- (2) *For every $\sigma \in \mathbb{R}$ the following Bernstein inequality holds true*

$$(4.19) \quad \|\Delta^\sigma f\| \leq \left(\omega^2 + \frac{1}{4} \right)^\sigma \|f\|,$$

where $\|f\|$ means the $L_2(\mathbb{H})$ norm of f .

- (3) *For every $g \in L_2(\mathbb{H}, d\mu)$ the function*

$$t \rightarrow \langle f * G_t, g \rangle = \int_{\mathbb{H}} f * G_t \bar{g} d\mu$$

is an entire function of the exponential type $\omega^2 + \frac{1}{4}$ bounded on the real line \mathbb{R} .

5. GENERALIZED SHANNON-TYPE SAMPLING IN PALEY-WIENER SPACES AND FRAMES IN L_2 -SPACES ON RIEMANNIAN MANIFOLDS OF BOUNDED GEOMETRY

In this section we treat both compact and non-compact Riemannian manifolds of bounded geometry whose Ricci curvature is bounded from below (see section 3.1). The material in this section is based on [104], [108].

5.1. A Shannon-type sampling theorem in Paley-Wiener spaces on compact and non-compact Riemannian manifolds. The most important fact for our development is an analogue of the Shannon's Sampling Theorem for Riemannian manifolds of bounded geometry which first appeared in [104] and was further developed in [39], [40], [107]-[118], (for subelliptic versions see [44], [46], [101], [102]).

Let \mathbf{M}_r be an r -lattice and let $\{B(x_k, r)\}$ be an associated family of balls that satisfy the properties of Definition 4.2. We define

$$U_1 = B(x_1, r/2) \setminus \cup_{i, i \neq 1} B(x_i, r/4),$$

and

$$(5.1) \quad U_k = B(x_k, r/2) \setminus (\cup_{j < k} U_j \cup_{i, i \neq k} B(x_i, r/4)).$$

It is easy to verify the following statement.

Lemma 5.1. *The sets $\{U_k\}$ form a disjoint measurable cover of \mathbf{M} and*

$$(5.2) \quad B(x_k, r/4) \subset U_k \subset B(x_k, r/2).$$

We assume that on every U_k a strictly positive measure μ_k , supported in U_k , is given. We consider the following distribution on $C_0^\infty(B(x_k, r))$,

$$(5.3) \quad \mathcal{M}_k(f) = \frac{1}{|U_k|_{\mu_k}} \int_{U_k} f d\mu_k, \quad \text{with} \quad |U_k|_{\mu_k} = \int_{U_k} d\mu_k, \quad f \in C_0^\infty(B(x_k, r)).$$

As a compactly supported distribution of order zero it has a unique continuous extension to a function on the space $C^\infty(B(x_k, r))$.

We say that a family $\mathcal{M} = \{\mathcal{M}_k\}$ is uniformly bounded, if there exists a positive constant $C_{\mathcal{M}}$ such that

$$(5.4) \quad \mathcal{M}_k(f) \leq C_{\mathcal{M}} \sup_{x \in B(x_k, r)} |f(x)|, \quad f \in C^\infty(B(x_k, r)) \quad \text{for all } k.$$

Some examples of distributions which are of particular interest are the following.

- (1) Weighted Dirac measures $\mathcal{M}_k(f) = a_k \delta_{x_k}(f)$, $x_k \in U_k$, $a_k > 0$.
- (2) Finite or infinite sequences of weighted Dirac measures

$$\mathcal{M}_k(f) = \sum_{x_{k,l} \in U_k} a_{k,l} \delta_{x_{k,l}}(f), \quad a_{k,l} > 0.$$

- (3) $d\mu_k$ is a "surface" measure on a submanifold contained in U_k .
- (4) $d\mu_k$ is the restriction to U_k of the Riemannian measure dx on \mathbf{M} .

Lemma 5.2. *(Local Poincaré-type inequality [108]) For $m > n/2$ ($n = \dim \mathbf{M}$) there exist positive constants $C = C(\mathbf{M}, m) > 0$, $r(\mathbf{M}, m) > 0$, such that for any $(r, N_{\mathbf{M}})$ -lattice M_r with $r < r(\mathbf{M}, m)$ and any associated family of functional \mathcal{M}_k the following inequality holds true for all $f \in H^m(\mathbf{M})$:*

$$(5.5) \quad \|(\varphi_\nu f) - \mathcal{M}_k((\varphi_\nu f))\|_{L_2(U_k)} \leq C \sum_{1 \leq |\alpha| \leq m} r^{|\alpha|} \|\partial^\alpha(\varphi_\nu f)\|_{L_2(B(x_k, r))},$$

where for any multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ the symbol $\partial^\alpha f$ stands for a partial derivative $\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ in a geodesic coordinate system.

We introduce the following set of functionals

$$(5.6) \quad \mathcal{A}_k(f) = \sqrt{|U_k|} \mathcal{M}_k(f) = \frac{\sqrt{|U_k|}}{|U_k|_{\mu_k}} \int_{U_k} f(x) d\mu_k, \quad |U_k| = \int_{U_k} dx, \quad f \in L_2(\mathbf{M}),$$

where dx is the Riemann measure on \mathbf{M} .

For the further development it will be important to associate with these functionals families of band-limited functions. In fact, the restriction of the functionals above to the closed subspaces $\mathbf{PW}_\omega(\sqrt{L}) \subset L_2(\mathbf{M})$ can be realized by scalar products, via the Riesz representation theorem: There are uniquely determined functions $\phi_{\omega,k} \in \mathbf{PW}_\omega(\sqrt{L})$ such that

$$(5.7) \quad \langle f, \phi_{\omega,k} \rangle = \mathcal{A}_{\omega,k}(f), \quad f \in \mathbf{PW}_\omega(\sqrt{L}).$$

It is convenient to introduce the following definition.

Definition 5.3. *For a given r -lattice \mathbf{M}_r , let $\{U_k\}$ be the disjoint cover constructed in (5.1), and functionals $\mathcal{M}_k, \mathcal{A}_k$ defined as in (5.3) and (5.6) respectively.*

Given an $\omega > 0$ the system of functions $\{\phi_{\omega,k}\}$ defined in (5.7) will be called the family of functions associated with the pair $(\mathbf{M}_r, \mathbf{PW}_\omega(\sqrt{L}))$.

Lemma 5.4. *(Global Poincaré-type inequality) For any $0 < \delta < 1$ and $m > \frac{1}{2} \dim \mathbf{M}$, there exist constants $c = c(\mathbf{M})$, $C = C(\mathbf{M}, m)$, such that following inequality holds true for any $(r, N_{\mathbf{M}})$ -lattice with $r \leq c\delta$ and any $H^m(\mathbf{M})$:*

$$(5.8) \quad (1 - \delta/2) \|f\|^2 \leq \sum_k |\mathcal{A}_k(f)|^2 + C\delta^{-1} r^{2m} \|L^{m/2} f\|^2.$$

Proof. We sketch a proof for the case of a manifold without boundary (compact or non-compact). In the presence of a boundary more care is required around the boundary. (see [132]). Applying Lemma 5.2 and the inequality

$$(5.9) \quad (1 - \alpha)|A|^2 \leq \frac{1}{\alpha}|A - B|^2 + |B|^2, \quad 0 < \alpha < 1,$$

we obtain

$$(5.10) \quad (1 - \delta/3) \|\varphi_\nu f\|_{L_2(U_k)}^2 \leq 3/\delta \sum_k \|f - \mathcal{M}_k(\varphi_\nu f)\|_{L_2(U_k)}^2 + \sum_k |U_k| |\mathcal{M}_k(\varphi_\nu f)|^2.$$

From this and (5.5) one obtains

$$(1 - \delta/3) \|f\|_{L_2(\mathbf{M})}^2 \leq \sum_\nu \sum_k |U_k| |\mathcal{M}_k(\varphi_\nu f)|^2 + C_1(M, m) \delta^{-1} \sum_{1 \leq j \leq m} r^{2j} \|f\|_{H^j(\mathbf{M})}^2.$$

The regularity theorem for the elliptic second-order differential operator L (see [66], Sec. 17.5)

$$(5.11) \quad \|f\|_{H^j(\mathbf{M})}^2 \leq b \left(\|f\|_2^2 + \|L^{j/2} f\|_2^2 \right), \quad f \in \mathcal{D}(L^{m/2}), \quad b = b(\mathbf{M}, j),$$

and the interpolation inequality (see [66], Sec. 17.5)

$$(5.12) \quad r^{2j} \|L^{j/2} f\|_2^2 \leq 4a^{m-j} r^{2m} \|L^{m/2} f\|_2^2 + ca^{-j} \|f\|_2^2, \quad c = c(\mathbf{M}, m),$$

which holds for any $a, r > 0$, $0 \leq j \leq m$, imply that there exists a constant $C = C(\mathbf{M}, m)$ such that for any $0 < \delta < 1$ and $r > 0$

$$(1 - \delta/3) \|f\|_2^2 \leq \sum_k |U_k| |\mathcal{M}_k(f)|^2 + C \left(r^2 \delta^{-1} \|f\|_2^2 + r^{2m} \delta^{-1} \|L^{m/2} f\|_2^2 \right).$$

The last inequality shows that if for a given $0 < \delta < 1$ and $c = (6C)^{-1/2}$ the value of r is chosen such that $r < c\delta$ then we obtain (5.8). The lemma is proved. \square

Let us pick $m = n = \dim \mathbf{M}$. With this choice the inequality (5.8) is the same as inequality (3.7) for $\rho_{\mathcal{A}} = r$ and $m_0 = n$. Note that for functions in $\mathbf{PW}_{\omega}(\sqrt{L})$ the Bernstein inequality holds

$$\|L^{m/2}f\|_2 \leq \omega^m \|f\|_2, \quad f \in \mathbf{PW}_{\omega}(\sqrt{L}).$$

Using property (5.4) and a Sobolev Embedding Theorem one can verify that condition (3.8) is satisfied. Thus Theorem 3.8 gives the following version of the Sampling Theorem. It describes the appropriate sampling density in relation to the ‘bandwidth’ of the PW -space (see [104], [108]):

Theorem 5.5. (*Almost Parseval frames in Paley-Wiener spaces*) *Let \mathbf{M} be a manifold of dimension n , with bounded geometry and Ricci curvature bounded from below. Then there exists $c = c(\mathbf{M}) > 0$ such that one has: Given $r > 0$ and $\omega > 0$ such that*

$$(5.13) \quad 0 < r \leq c \delta^{1/n} \omega^{-1},$$

then any family of functions $\{\phi_{\omega,k}\}$ associated to the pair $(\mathbf{M}_r, \mathbf{PW}_{\omega}(\sqrt{L}))$ (see Definition 5.3) forms a frame in $\mathbf{PW}_{\omega}(\sqrt{L})$, and the following Plancherel-Polya-type inequalities (frame inequalities) hold true:

$$(5.14) \quad (1 - \delta) \|f\|_2^2 \leq \sum_k |\langle f, \phi_{\omega,k} \rangle|^2 \leq \|f\|_2^2, \quad f \in \mathbf{PW}_{\omega}(\sqrt{L}).$$

5.2. Methods of reconstruction of Paley-Wiener functions. For reconstruction of a function from the set of samples one can use besides dual frames the following methods:

- (1) Reconstruction by variational (polyharmonic) splines on manifolds [101]-[118].
- (2) Reconstruction using iterations [39], [40].
- (3) Reconstruction by the frame algorithm [61].

5.3. Optimality of the number of sampling points on compact manifolds.

The material in this section is based on [108], [118]. Condition (5.13) imposes a specific rate of sampling. It is interesting to note that this rate is essentially optimal. Indeed, since \sqrt{L} is a non-negative elliptic pseudodifferential operator of order one the Weyl’s asymptotic formula [66] gives

$$(5.15) \quad \mathcal{N}_{\omega}(\sqrt{L}) \asymp C \text{Vol}(\mathbf{M}) \omega^n,$$

where $\mathcal{N}_{\omega}(\sqrt{L})$ is the dimension of the space $\mathbf{PW}_{\omega}(\sqrt{L})$ and $\text{Vol}(\mathbf{M})$ is the volume of \mathbf{M} . On the other hand, condition (5.13) and the definition of a r -lattice imply that the number of points in an ”optimal” lattice M_r is approximately

$$\text{card } M_r \sim \frac{\text{Vol}(\mathbf{M})}{c_0^n \omega^{-n}} = c \text{Vol}(\mathbf{M}) \omega^n, \quad n = \dim \mathbf{M},$$

which is consistent with Weyl’s formula. Note that the power n in this formula, which is what one would expect from the Shannon sampling theorem in the euclidean setting, results from our use of the spectrum of \sqrt{L} rather than that of L .

5.4. Paley-Wiener almost Parseval frames in L_2 spaces on compact and non-compact Riemannian manifolds. We return to the notation of Subsections 3.4 and 3.5. Let $\omega_j = 2^{j+1}$, for $j \geq 0$. According to Theorem 5.5 for a fixed $0 < \delta < 1$ there exists a constant $c = c(\mathbf{M})$ such that for

$$r_j = c\delta^{1/n}\omega_j^{-1} = c\delta^{1/n}2^{-j-1}, \quad j \in \mathbb{N},$$

and any r_j -lattice $M_{r_j} = \{x_{j,k}\}$, $1 \leq k \leq \mathcal{K}_j$, the inequalities (5.14) hold.

For every $j \in \mathbb{N}$ let $\mathcal{M}_j = \{\mathcal{M}_{j,k}\}$ be an associated set of distributions described in (5.3). For the sake of simplicity we now assume that for every $j \in \mathbb{N}$ the set of distributions $\mathcal{M}_j = \{\mathcal{M}_{j,k}\}$ described in (5.3) consists of Dirac measures $\delta_{j,k}$ at points $x_{j,k}$. Let $\{\mathcal{A}_k^j\}$ be the corresponding set of functionals defined in (5.6) and let $\phi_k^j \in \mathbf{PW}_{\omega_j}(\sqrt{L}) = \mathbf{PW}_{2^{j+1}}(\sqrt{L})$ be a function such that

$$(5.16) \quad \langle f, \phi_k^j \rangle = \mathcal{A}_k^j(f),$$

for all $f \in \mathbf{PW}_{2^{j+1}}(\sqrt{L})$. If $(F_j)_{j \geq 0}$ is the quadratic partition of unity introduced in Subsection 3.4 then since $F_j(\sqrt{L})f \in \mathbf{PW}_{2^{j+1}}(\sqrt{L})$ we have according to (5.14) the following frame inequalities for every $j \in \mathbb{Z}$

$$(5.17) \quad (1 - \delta) \left\| F_j(\sqrt{L})f \right\|_2^2 \leq \sum_{k=1}^{\mathcal{K}_j} \left| \langle F_j(\sqrt{L})f, \phi_k^j \rangle \right|^2 \leq \left\| F_j(\sqrt{L})f \right\|_2^2, \quad f \in L_2(\mathbf{M}).$$

But since the operator $F_j(\sqrt{L})$ is self-adjoint, we obtain (via (3.16)) that for the functions

$$(5.18) \quad \Phi_k^j = F_j(\sqrt{L})\phi_k^j$$

which are bandlimited to $[2^{j-1}, 2^{j+1}]$, the following frame inequalities hold

$$(5.19) \quad (1 - \delta) \|f\|_2^2 \leq \sum_{j \geq 0} \sum_{k=1}^{\mathcal{K}_j} \left| \langle f, \Phi_k^j \rangle \right|^2 \leq \|f\|_2^2, \quad f \in L_2(\mathbf{M}).$$

To summarize, let us assume that a Riemannian manifold \mathbf{M} has bounded geometry and (4.3) holds. Let $c > 0$ be a positive constant and $0 < \delta < 1$. We consider the following:

- (1) a sequence of r_j -lattices $M_{r_j} = \{x_{j,k}\}$, $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, with

$$r_j = c\delta^{1/n}2^{-j-1}, \quad j \geq 0,$$

- (2) a set of disjoint coverings $\{U_k^j\}$, $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, as in (5.1),
(3) a set of functionals \mathcal{A}_k^j , $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, defined as in (5.6),
(4) a set of functions ϕ_k^j , $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, defined as in (5.16),
(5) a set of functions Φ_k^j in $\mathbf{PW}_{[2^{j-1}, 2^{j+1}]}(\sqrt{L})$, $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, as in (5.18).

In this notation Theorem 3.9 takes the following form:

Theorem 5.6. *(Paley-Wiener frames in $L_2(\mathbf{M})$) Suppose a Riemannian manifold \mathbf{M} has bounded geometry and (4.3) holds. Then there exists a constant $c = c(\mathbf{M})$ such that for any $0 < \delta < 1$ the set of functions Φ_k^j , $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, defined in (5.18) has the following properties:*

- (1) $\Phi_k^j \in \mathbf{PW}_{[2^{j-1}, 2^{j+1}]}(\sqrt{L})$, $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$;

(2) the family $\{\Phi_k^j\}$ is a frame in \mathcal{H} with constants $1 - \delta$ and 1:

$$(5.20) \quad (1 - \delta)\|f\|_2^2 \leq \sum_{j \geq 0} \sum_k \left| \langle f, \Phi_k^j \rangle \right|^2 \leq \|f\|_2^2, \quad f \in \mathcal{H}.$$

(3) the canonical dual frame $\{\Psi_k^j\}$ is also bandlimited with

$$\Psi_k^j \in \mathbf{PW}_{[2^{j-1}, 2^{j+1}]}(\sqrt{L}), \quad j \in \mathbb{N}, \quad 1 \leq k \leq \mathcal{K}_j;$$

and satisfies the inequalities

$$(5.21) \quad \|f\|_2^2 \leq \sum_{j \geq 0} \sum_k \left| \langle f, \Psi_k^j \rangle \right|^2 \leq (1 - \delta)^{-1} \|f\|_2^2, \quad f \in \mathcal{H}.$$

(4) the reconstruction formulas hold for every $f \in \mathcal{H}$

$$(5.22) \quad f = \sum_j \sum_k \langle f, \Phi_k^j \rangle \Psi_k^j = \sum_j \sum_k \langle f, \Psi_k^j \rangle \Phi_k^j.$$

6. ALMOST PARSEVAL SPACE-FREQUENCY LOCALIZED FRAMES ON GENERAL COMPACT RIEMANNIAN MANIFOLDS

In this section we consider only compact Riemannian manifolds.

6.1. Kernels on compact manifolds. In the situation of a compact manifold, let \sqrt{L} be the positive square root of a second order differential elliptic selfadjoint nonnegative operator L in $L_2(\mathbf{M})$. Let $0 = \lambda_0 < \lambda_1 \leq \dots$ be the spectrum of \sqrt{L} , and $\{u_l\}$ denote a sequence of corresponding eigenfunctions which is an orthonormal basis in $L_2(\mathbf{M})$. In this case formulas (3.2) and (2.1) correspond to

$$(6.1) \quad F(\sqrt{L})f(x) = \sum_{\lambda_l} F(\lambda_l) c_l(f) u_l(x), \quad F \in C_0^\infty(\mathbb{R}).$$

For any $t > 0$ one defines a bounded operator by the formula

$$(6.2) \quad F(t\sqrt{L})f(x) = \int_{\mathbf{M}} K_t^F(x, y) f(y) dy = \langle K_t^F(x, \cdot), f(\cdot) \rangle,$$

where

$$K_t^F(x, y) = \sum_l F(t\lambda_l) u_l(x) u_l(y) = K_t^F(y, x).$$

We call K_t^F the kernel of the operator $F(t\sqrt{L})$. This operator maps $C^\infty(\mathbf{M})$ to itself continuously, and may thus be extended to be a map on distributions. In particular we may apply $F(t\sqrt{L})$ to any $f \in L_p(\mathbf{M}) \subseteq L_1(\mathbf{M})$ (where $1 \leq p \leq \infty$), and by Fubini's theorem $F(t\sqrt{L})f$ is still given by (6.2).

It is quite obvious that for a fixed $x \in \mathbf{M}$ the kernel $K_t^F(x, y)$ is approaching the Dirac measure δ_x in the sense of distributions when $t > 0$ goes to zero. However, an explicit pointwise estimate is needed.

In this situation the following analogue of the estimate (2.4) can be proved (see [57]) using the language of pseudodifferential operators.

Theorem 6.1. *Assume that F belongs to $C_0^\infty(\mathbb{R}_+)$, $\text{supp } F \subset [0, \rho]$ with $\rho > 1$, and $F^{(k)}(0) = 0$ for all odd $k \geq 1$. Let $K_t^F(x, y)$ be the kernel of $F(t\sqrt{L})$.*

For any $N > n = \dim \mathbf{M}$ there exists a constant $C = C(F, N)$ such that the following inequality holds true for $0 < t \leq 1$:

$$(6.3) \quad |K_t^F(x, y)| \leq \frac{C}{t^n} \left[\left(1 + \frac{d(x, y)}{t} \right) \right]^{-N}, \quad x, y \in \mathbf{M}.$$

Here $0 < C = C(F, N) \leq c_N \|F\|_{C^N[0, \rho]} \rho^N$ for some c_N which depends only on N .

6.2. Almost Parseval space-localized Paley-Wiener frames on general compact manifolds. We return to nearly Parseval Paley-Wiener frames which were described in Theorem 5.6 and we make additional assumption that \mathbf{M} is compact.

Recall for the following theorem that we consider a specific constant $c = c(\mathbf{M}) > 0$ and for a fixed $0 < \delta < 1$ introduce a sequence of r_j -lattices $M_{r_j} = \{x_{j,k}\}$, $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, with

$$r_j = c\delta^{1/n}2^{-j-1}, \quad j \geq 0.$$

a set of disjoint coverings $\{U_k^j\}$, $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, as in (5.1),

a set of functionals \mathcal{A}_k^j , $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, defined as in (5.6),

a set of functions ϕ_k^j , $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, defined as in (5.16),

a set of functions $\Phi_k^j \in \mathbf{PW}_{[2^{j-1}, 2^{j+1}]}(\sqrt{L})$, $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, as in (5.18).

Since

$$\langle f, \Phi_k^j \rangle = \langle f, F_j(\sqrt{L})\phi_k^j \rangle = \langle F_j(\sqrt{L})f, \phi_k^j \rangle = \mathcal{A}_k^j(F_j(\sqrt{L})f)$$

we have the following explicit formula

$$(6.4) \quad \Phi_k^j(y) = \mathcal{A}_k^j(K_{2^{-j}}(x, y)) = \frac{\sqrt{|U_{j,k}|}}{|U_{j,k}|_{\mu_k}} \int_{U_{j,k}} K_{2^{-j}}(x, y) d\mu_{j,k}(x),$$

where

$$|U_{j,k}|_{\mu_k} = \int_{U_{j,k}} d\mu_{j,k}, \quad |U_{j,k}| = \int_{U_{j,k}} dx,$$

dx being the Riemann measure on \mathbf{M} . Note that according to (6.3) each Φ_k^j satisfies the following estimate for $N > \dim \mathbf{M} = n$:

$$(6.5) \quad \left| \Phi_k^j(y) \right| = \sqrt{|U_{j,k}|} \sup_{x \in U_{j,k}} |K_{2^{-j}}(x, y)| \leq C(N) \sup_{x \in U_{j,k}} \frac{2^{j(n-N)}}{(2^{-j} + d(x, y))^N}.$$

Using this notation Theorem 3.9 takes the following form.

Theorem 6.2. (Paley-Wiener frames in $L_2(\mathbf{M})$) For any compact Riemannian manifold \mathbf{M} there exists a constant $c = c(\mathbf{M})$ such that for any $0 < \delta < 1$ the set of functions Φ_k^j , $j \in \mathbb{N}$, $1 \leq k \leq \mathcal{K}_j$, defined in (5.18) has the following properties:

- (1) each function Φ_k^j belongs to $\mathbf{PW}_{[2^{j-1}, 2^{j+1}]}(\sqrt{L})$, $j \in \mathbb{N}$, $k = 1, \dots$;
- (2) each Φ_k^j is localized according to (6.5);
- (3) the family $\{\Phi_k^j\}$ is a frame in \mathcal{H} with constants $1 - \delta$ and 1:

$$(6.6) \quad (1 - \delta) \|f\|_2^2 \leq \sum_{j \geq 0} \sum_k \left| \langle f, \Phi_k^j \rangle \right|^2 \leq \|f\|_2^2, \quad f \in \mathcal{H};$$

- (4) the canonical dual frame $\{\Psi_k^j\}$ is also bandlimited, i.e. $\Psi_k^j \in \mathbf{PW}_{[2^{j-1}, 2^{j+1}]}(\sqrt{L})$, $j \in [0, \infty)$, $k = 1, \dots$, and satisfies the inequalities

$$(6.7) \quad \|f\|_2^2 \leq \sum_{j \geq 0} \sum_k \left| \langle f, \Psi_k^j \rangle \right|^2 \leq (1 - \delta)^{-1} \|f\|_2^2, \quad f \in \mathcal{H};$$

- (5) the reconstruction formulas hold for every $f \in \mathcal{H}$:

$$(6.8) \quad f = \sum_j \sum_k \langle f, \Phi_k^j \rangle \Psi_k^j = \sum_j \sum_k \langle f, \Psi_k^j \rangle \Phi_k^j.$$

7. LITTLEWOOD-PALEY DECOMPOSITION IN L_p SPACES ON COMPACT MANIFOLDS

In this subsection we extend the Littlewood-Paley decompositions to the L^p -setting. The ideas in this section are somewhat similar to [8].

7.1. More about kernels. The estimate (6.3) has an important implication:

Corollary 7.1. *For $F \in C_0^\infty(\mathbb{R}_+)$ as in Theorem 6.1 and $1 \leq p \leq \infty$ there exists a constant $c = c(F, p) > 0$ such that with $n = \dim \mathbf{M}$, $1/p + 1/q = 1$ one has:*

$$(7.1) \quad \left(\int_{\mathbf{M}} |K_t^F(x, y)|^p dy \right)^{1/p} \leq ct^{-n/q}, \quad \text{for all } 0 < t \leq 1, x \in \mathbf{M},$$

Proof. It is enough to show that for $N > n = \dim \mathbf{M}$ there exists a $C(N) > 0$ such that one has:

$$(7.2) \quad \int_{\mathbf{M}} \frac{1}{[1 + (d(x, y)/t)]^N} dy \leq C(N)t^n, \quad \text{for all } x \in \mathbf{M}, t > 0.$$

Indeed, there exist $c_1, c_2 > 0$ such that for all sufficiently small $r \leq \delta$ one has

$$c_1 r^n \leq |B(x, r)| \leq c_2 r^n, \quad \text{for all } x \in M$$

and if $r > \delta$

$$c_3 \delta^n \leq |B(x, r)| \leq |\mathbf{M}| \leq c_4 r^n \quad \text{for all } x \in M.$$

For fixed x, t set $A_j = B(x, 2^j t) \setminus B(x, 2^{j-1} t)$. Then $|A_j| \leq c_4 2^{nj} t^n$ and one has

$$\begin{aligned} & \int_{\mathbf{M}} \frac{1}{[1 + (d(x, y)/t)]^N} dy = \\ & \sum_j \int_{A_j} \frac{1}{[1 + (d(x, y)/t)]^N} dy \leq c_4 2^N \sum_j 2^{j(n-N)} t^n \leq C(N) t^n. \end{aligned}$$

Using this estimate and (6.3) for $N = n + 1$ one obtains (7.1) for $p < \infty$. The case $p = \infty$ is obvious. \square

Theorem 7.1. *Let F be as in Theorem 6.1 and $(1/q) + 1 = (1/p) + (1/\alpha)$. For the same constant c as in (7.1) one has for all $0 < t \leq 1$:*

$$\|F(t\sqrt{L})\|_{L_p(\mathbf{M}) \rightarrow L_q(\mathbf{M})} \leq ct^{-n/\alpha'}, \quad 1/\alpha + 1/\alpha' = 1, \quad n = \dim \mathbf{M}.$$

In particular, one has for $\alpha = 1$

$$\|F(t\sqrt{L})\|_{L_p(\mathbf{M}) \rightarrow L_p(\mathbf{M})} \leq c.$$

Proof. The proof follows from Corollary 7.1 and the following Young inequality:

Lemma 7.2. *Let $\mathcal{K}(x, y)$ be a measurable function on $\mathbf{M} \times \mathbf{M}$. Given $1 \leq p, \alpha \leq \infty$, we set $(1/q) + 1 = (1/p) + (1/\alpha)$. If there exists a $C > 0$ such that*

$$(7.3) \quad \left(\int_{\mathbf{M}} |\mathcal{K}(x, y)|^\alpha dy \right)^{1/\alpha} \leq C \quad \text{for all } x \in \mathbf{M},$$

and

$$(7.4) \quad \left(\int_{\mathbf{M}} |\mathcal{K}(x, y)|^\alpha dx \right)^{1/\alpha} \leq C \quad \text{for all } y \in \mathbf{M},$$

then one has for the same constant C the inequality

$$\left\| \int_{\mathbf{M}} \mathcal{K}(x, y) f(y) dy \right\|_q \leq C \|f\|_p \quad f \in L_p(\mathbf{M}).$$

□

7.2. Littlewood-Paley decomposition. For this subsection we recall the dyadic partition $(G_j)_{j \geq 0}$ from Subsection 3.4. In particular, we have $\sum_{j=0}^{\infty} G_j(\lambda) = 1$ for every $\lambda \geq 0$, as well as $\text{supp}(G_j) \subset [2^{j-1}, 2^{j+1}]$ for $j \geq 1$.

Lemma 7.3. *For $m \in \mathbb{N}$ there exists a $C > 0$ such that*

$$(7.5) \quad \left\| G_j(\sqrt{L}) f \right\|_q \leq C \left(2^{(j-1)n} \right)^{-\frac{m}{n} + \frac{1}{p} - \frac{1}{q}} \|f\|_{W_p^m(\mathbf{M})}, \quad n = \dim \mathbf{M},$$

for all $f \in W_p^m(\mathbf{M})$. In other words, the norm of $G_j(\sqrt{L})$, as an element of the space $\mathbf{B}(W_p^m(\mathbf{M}), L_q(\mathbf{M}))$ of bounded linear operators from $W_p^m(\mathbf{M})$ to $L_q(\mathbf{M})$, is bounded by $C \left(2^{(j-1)n} \right)^{-\frac{m}{n} + \frac{1}{p} - \frac{1}{q}}$.

Proof. For $\lambda > 0$ we set $\Psi(\lambda) = G_1(\lambda) \lambda^{-m}$. Consequently Ψ is supported in $[1, 4]$. For $j \geq 1$, we set

$$\Psi_j(\lambda) = \Psi \left(2^{-(j-1)} \lambda \right) = 2^{(j-1)m} G_j(\lambda) \lambda^{-m},$$

so that

$$G_j(\lambda) = 2^{-(j-1)m} \Psi_j(\lambda) \lambda^m.$$

Accordingly, if f is a distribution on \mathbf{M} , for $j \geq 1$, one has

$$G_j(\sqrt{L}) f = 2^{-(j-1)m} \Psi_j(\sqrt{L}) \left(L^{m/2} f \right),$$

in the sense of distributions. For $f \in W_p^m(\mathbf{M})$ one has $L^{m/2} f \in L_p(\mathbf{M})$, and by Theorem 7.1 we obtain that if $(1/q) + 1 = (1/p) + (1/\alpha)$ and $1/\alpha + 1/\alpha' = 1$ then

$$\left\| G_j(\sqrt{L}) f \right\|_q \leq C 2^{-(j-1)m} 2^{(j-1)n/\alpha'} \|L^{m/2} f\|_p \leq C \left(2^{(j-1)n} \right)^{-\frac{m}{n} + \frac{1}{p} - \frac{1}{q}} \|f\|_{W_p^m(\mathbf{M})}.$$

□

Using the same notation we formulate the following result (see [59]).

Theorem 7.4. *(Littlewood-Paley decomposition) The series $\sum_{j=0}^{\infty} G_j(\sqrt{L})$ converges strongly to the identity operator, i.e. one has in the norm of $L_p(\mathbf{M})$:*

$$(7.6) \quad \sum_{j=0}^{\infty} G_j(\sqrt{L}) f = f, \quad f \in L_p(\mathbf{M}).$$

Proof. By the previous lemma the series $\sum_{j=0}^{\infty} G_j(\sqrt{L})$ converges in the norm of $\mathbf{B}(W_p^m(\mathbf{M}), L_p(\mathbf{M}))$. It converges to the identity on smooth functions, hence in the sense of distributions. Therefore we must have $\sum_{j=0}^{\infty} G_j(\sqrt{L}) = I$ in $\mathbf{B}(W_p^m(\mathbf{M}), L_p(\mathbf{M}))$. Hence we get strong convergence on a dense subspace, and it will be sufficient to verify the uniform boundedness of the operator norms in $\mathbf{B}(L_p(\mathbf{M}), L_p(\mathbf{M}))$ in order to finish the proof. However, this follows from (3.13) and Theorem 7.1, applied to g . \square

8. LOCALIZED PARSEVAL FRAMES IN $L_2(\mathbf{M})$ OVER COMPACT HOMOGENEOUS MANIFOLDS

8.1. Product property for eigenfunctions of the Casimir operator on homogeneous compact manifolds. The following important theorem was proved in [57], [124] and it is necessary for the construction of Parseval frames on a homogeneous compact manifold. Note, that this Theorem is an analog of the Lemma 2.4.

Theorem 8.1. (*Product property on homogeneous manifolds*) Let $\mathbf{M} = G/K$ be a compact homogeneous manifold and \mathcal{L} as in (4.12). Then for any $f, g \in \mathbf{E}_\omega(\sqrt{\mathcal{L}})$ the pointwise product fg is in $\mathbf{E}_{4d\omega}(\sqrt{\mathcal{L}})$, where d is the dimension of the group G .

Proof. The proof is using the following two lemmas from [115].

Lemma 8.2. For a self-adjoint operator A in a Hilbert space \mathcal{H} vector f belongs to a Paley-Wiener space $\mathbf{PW}_\omega(A)$, $\omega > 0$, if and only if there exists a constant $C = C(f, \omega)$ such that for all natural k the following Bernstein-type inequality holds $\|A^k f\| \leq C(f, \omega)\omega^k$.

Lemma 8.3. For D_1, \dots, D_d , $d = \dim G$, as in (4.11) then the following equalities hold

$$\|\mathcal{L}^{k/2} f\|_2^2 = \sum_{1 \leq i_1, \dots, i_k \leq d} \|D_{i_1} \dots D_{i_k} f\|_2^2, \quad k \in \mathbb{N}.$$

Remark 8.4. In the case of torus \mathbb{T}^n and \mathbb{R}^n where $D_j = \frac{\partial}{\partial x_j}$ this statement can be easily proved by using Fourier transform.

Next, one shows that for any smooth functions f, g the following estimate holds

$$|\mathcal{L}^k(fg)| \leq (4d)^k \sup_{0 \leq m \leq 2k} \sup_{x, y \in \mathbf{M}} |D_{i_1} \dots D_{i_m} f(x)| |D_{j_1} \dots D_{j_{2k-m}} g(y)|.$$

From here by using Lemma 8.3, the Sobolev embedding theorem and elliptic regularity of \mathcal{L} , one obtains the estimate

$$(8.1) \quad \|\mathcal{L}^k(fg)\|_2 \leq C(\mathbf{M}, f, g, \omega)(4d\omega)^k, \quad k \in \mathbb{N},$$

which according to Lemma 8.2 implies Theorem 8.1. \square

EXAMPLE. For the choice $\mathbf{M} = \mathbf{S}$, the unit circle, the Laplace-Beltrami operator is $\mathcal{L} = \left(\frac{d}{d\varphi}\right)^2$ whose real eigenfunctions are $\sin k\varphi$, $\cos m\varphi$, $k, m \in \mathbb{N}$. In this case the following identities illustrate our theorem:

$$\sin k\varphi \cos m\varphi = \frac{1}{2} \sin(k+m)\varphi + \frac{1}{2} \cos(k-m)\varphi$$

$$\sin^2 k\varphi = \frac{1}{2}(1 - \cos 2k\varphi), \quad \cos^2 k\varphi = \frac{1}{2}(1 + \cos 2k\varphi)$$

At this moment it is not known if the constant $4d$ can be lowered in the general situation. However, it is possible to verify that in the cases of a torus, a sphere or of a projective space of any dimension the best constant is 2:

$$f, g \in \mathbf{E}_\omega(\mathcal{L}) \rightarrow fg \in \mathbf{E}_{2\omega}(\mathcal{L}).$$

8.2. Positive cubature formulas on manifolds. Now we are going to formulate a result about the existence of *cubature formulas* which are exact on $\mathbf{E}_\omega(\sqrt{L})$, and have positive coefficients of the "right" size. The following exact cubature formula was established in [57], [124].

Theorem 8.5. *If \mathbf{M} is a compact Riemannian manifold then there exists a positive constant $a = a(\mathbf{M})$, such that for $0 < r < a\omega^{-1}$, for any r -lattice $\mathbf{M}_r = \{x_k\}$ there exist strictly positive coefficients $a_{x_k} > 0$, $x_k \in \mathbf{M}_r$ for which the following equality holds for all functions in $\mathbf{E}_\omega(\sqrt{L})$:*

$$(8.2) \quad \int_{\mathbf{M}} f dx = \sum_{x_k \in M_r} a_{x_k} f(x_k).$$

Moreover, there exists constants c_1, c_2 , such that the following inequalities hold:

$$(8.3) \quad c_1 r^n \leq a_{x_k} \leq c_2 r^n, \quad n = \dim \mathbf{M} \text{ for all } k.$$

8.3. Parseval frames on compact homogeneous manifolds. We again recall the quadratic partition of unity $(F_j)_{j \geq 0}$ constructed in subsection (3.4).

Since for every $\overline{F_j(\sqrt{L})}f \in \mathbf{PW}_{2^{j+1}}(\sqrt{L})$ one can use the product property (Theorem 8.1) to conclude that

$$\left| F_j(\sqrt{L})f \right|^2 \in \mathbf{PW}_{4d2^{j+1}}(\sqrt{L}),$$

where $d = \dim G$, $\mathbf{M} = G/H$. This shows that for every $f \in L_2(\mathbf{M})$ we have the following decomposition

$$(8.4) \quad \sum_{j \geq \infty} \left\| F_j(\sqrt{L})f \right\|_2^2 = \|f\|_2^2, \quad \left| F_j(\sqrt{L})f \right|^2 \in \mathbf{PW}_{4d2^{j+1}}(\sqrt{L}).$$

According to our cubature formula (Theorem 8.5) there exists a constant $a > 0$ such that for

$$(8.5) \quad r_j = a \cdot 4d \cdot 2^{-(j+1)} \asymp 2^{-j}, \quad d = \dim G, \quad \mathbf{M} = G/H,$$

and corresponding r_j -lattice $M_{r_j} = \{x_{j,k}\}$ one can find coefficients $a_{j,k}$ with $a_{j,k} \asymp r_j^n$, $n = \dim \mathbf{M}$, for which the following exact cubature formula holds

$$(8.6) \quad \left\| F_j(\sqrt{L})f \right\|_2^2 = \sum_{k=1}^{\mathcal{K}_j} a_{j,k} \left| F_j(\sqrt{L})f(x_{j,k}) \right|^2,$$

where $\mathcal{K}_j = \text{card}(\mathbf{M}_{r_j})$. Using the kernel $K_{2^{-j}}^F$ of the operator $F_j(\sqrt{L})$ we define

$$(8.7) \quad \Theta_k^j(y) = \sqrt{a_{j,k}} \overline{K_{2^{-j}}^F}(x_{j,k}, y) = \sqrt{a_{j,k}} \sum_{\lambda_m \in [2^{j-1}, 2^{j+1}]} \overline{F}(2^{-j}\lambda_m) \overline{u}_m(x_{j,k}) u_m(y).$$

Consequently we have the following equality

$$\|f\|_2^2 = \sum_{j,k} |\langle f, \Theta_k^j \rangle|^2 \quad \text{for all } f \in L_2(\mathbf{M}).$$

This, together with Theorem 6.1, gives the following statement (see [57]).

Theorem 8.6. (*Parseval frames on homogeneous manifolds*) *For any compact homogeneous manifold \mathbf{M} the family of functions $\{\Theta_k^j\}$ constructed in (8.3) forms a Parseval frame in the Hilbert space for $L_2(\mathbf{M})$. In particular the following reconstruction formula holds true*

$$(8.8) \quad f = \sum_{j \geq 0} \sum_{k=1}^{\mathcal{K}_j} \langle f, \Theta_k^j \rangle \Theta_k^j, \quad f \in L_2(\mathbf{M}).$$

For any $j \geq 1$ the functions Θ_k^j are bandlimited to $[2^{j-1}, 2^{j+1}]$, and for every $N > 0$ there exists a constant $C(N)$ such that with $n = \dim \mathbf{M}$ one has

$$(8.9) \quad |\Theta_k^j(x)| \leq C(N) \frac{2^{j(n-N)}}{(2^{-j} + d(x, x_{j,k}))^N}.$$

for all natural j .

8.4. An exact discrete formula for evaluating Fourier coefficients on compact homogeneous manifolds. As an application of the Product Property and the Cubature Formula, we formulate the following theorem which shows that in the case of a compact homogeneous manifold \mathbf{M} , for any fixed bandwidth ω , one can find finite sets of points which yield exact discrete formulas for computing Fourier coefficients of all bandlimited functions of bandwidth ω (see [124]).

Theorem 8.7. *For every compact homogeneous manifold $\mathbf{M} = G/H$ there exists a constant $c = c(\mathbf{M})$ such that for any $\omega > 0$ and any lattice $\mathbf{M}_r = \{x_k\}_{k=1}^{K_\omega}$ with $0 < r < c\omega^{-1}$ one can find positive weights a_k comparable to ω^{-n} , $n = \dim \mathbf{M}$, such that Fourier coefficients $c_j(f)$ of any $f \in \mathbf{E}_\omega(\sqrt{L})$ with respect to the basis $\{u_j\}_{j=1}^\infty$ can be computed by the following exact formula*

$$(8.10) \quad c_j(f) = \int_{\mathbf{M}} f(x) \overline{u_j}(x) dx = \sum_{k=1}^{K_\omega} a_k f(x_k) \overline{u_j}(x_k),$$

with r_ω satisfying relations

$$(8.11) \quad C_1(\mathbf{M})\omega^n \leq K_\omega \leq C_2(\mathbf{M})\omega^n.$$

We also have a discrete representation formula using the eigenfunctions u_j

$$(8.12) \quad f = \sum_j \sum_{k=1}^{K_\omega} a_k f(x_k) \overline{u_j}(x_k) u_j \quad \text{for all } f \in \mathbf{E}_\omega(L).$$

9. APPROXIMATION IN L_p NORMS AND BESOV SPACES ON GENERAL COMPACT MANIFOLDS

In Section 3.7 Besov subspaces in an abstract Hilbert space \mathcal{H} were characterized in terms of approximation by Paley-Wiener vectors, which were defined using an selfadjoint operator L . In particular, if $\mathcal{H} = L_2(\mathbf{M})$, for a Riemannian manifold \mathbf{M} of bounded geometry (compact or non-compact), and L is the positive square

root of a non-negative self-adjoint second order elliptic C^∞ -bounded differential operator, one obtains characterizations of the Besov spaces $\mathcal{B}_{2,p}^\alpha(\mathbf{M})$ in terms of approximation by Paley-Wiener functions on \mathbf{M} . The goal of this section is to develop the approximation theory for the spaces $L_p(\mathbf{M})$, $1 \leq p \leq \infty$ when \mathbf{M} is a compact Riemannian manifold. For more details see also [57], [59], [115].

9.1. The L_p -Jackson inequality. For $1 \leq p \leq \infty$ and f in $L_p(\mathbf{M})$, we set

$$(9.1) \quad \mathcal{E}(f, \omega, p) = \inf_{g \in \mathbf{E}_\omega(L)} \|f - g\|_p.$$

Lemma 9.1. *For every $m \in \mathbb{N}$ and $1 \leq p \leq \infty$ there exists a constant $C = C(\mathbf{M}, m, p)$ such that for any $\omega > 1$ and all $f \in W_p^m(\mathbf{M})$*

$$\mathcal{E}(f, \omega, p) \leq C\omega^{-m} \|L^{m/2} f\|_p.$$

Proof. Here we make use of the partition of unity $(G_j)_{j \geq 0}$ defined in Subsection 3.4. Recall that $\sum_{j \geq 0} G_j(\lambda) = 1$ for all $\lambda \in [0, \infty)$, and $\text{supp} G_j \subset [2^{j-1}, 2^{j+1}]$, for all $j \geq 1$. Furthermore, we have $G_j(\lambda) = G_1(2^{-j}\lambda)$, and $G_1(\lambda) = g(\lambda/2) - g(\lambda)$, for a suitably chosen function g with support in $[0, 2]$. We define for $\lambda > 0$

$$\Psi(\lambda) = G_1(\lambda)\lambda^{-m}$$

so that Ψ is supported in $[1, 4]$. For $j \geq 1$, we set

$$\Psi_j(\lambda) = \Psi(2^{-(j-1)}\lambda) = 2^{(j-1)m} G_j(\lambda)\lambda^{-m},$$

so that

$$G_j(\lambda) = 2^{-(j-1)m} \Psi_j(\lambda)\lambda^m.$$

Now for a given ω we change the variable λ to the variable $2\lambda/\omega$. Clearly, the support of $g(2\lambda/\omega)$ is the interval $[0, \omega]$ and we have the following relation

$$G_j(2\lambda/\omega) = \omega^{-m} 2^{-(j-1)m} \Psi_j(2\lambda/\omega)\lambda^m.$$

It implies that if $f \in W_p^m(\mathbf{M})$, so that $L^{m/2} f \in L_p(\mathbf{M})$ we have

$$G_j(2\sqrt{L}/\omega) f = \omega^{-m} 2^{-(j-1)m} \Psi_j(2\sqrt{L}/\omega) L^{m/2} f.$$

According to Theorem 7.1 we have the estimate

$$\|\Psi_j(2\sqrt{L}/\omega) u\|_p \leq C' \|u\|_p, \quad u \in L_p(\mathbf{M}).$$

Note, that since $g(2\lambda/\omega)$ has support in $[0, 2]$ the function $g(2\sqrt{L}/\omega) f$ belongs to $\mathbf{E}_\omega(\sqrt{L})$.

Thus, the last two formulas imply the following final inequality

$$\begin{aligned} \mathcal{E}(f, \omega, p) &\leq \left\| f - g(2\sqrt{L}/\omega) f \right\|_p \leq \sum_{j \geq 1} \left\| G_j(2\sqrt{L}/\omega) f \right\|_p \leq \\ &C' \omega^{-m} \sum_{j \geq 1} 2^{-(j-1)m} \|L^{m/2} f\|_p \leq C \omega^{-m} \|L^{m/2} f\|_p. \end{aligned}$$

The proof is complete. \square

9.2. The L_p -Bernstein inequality.

Lemma 9.2. *Given $m \in \mathbb{N}$ and $1 \leq p \leq \infty$ there exists a constant $C = C(\mathbf{M}, m, p) > 0$ such that for any $\omega > 1$ and all $f \in \mathbf{E}_\omega(\sqrt{L})$*

$$(9.2) \quad \|L^{m/2}f\|_p \leq C\omega^m \|f\|_p \quad \text{for all } f \in \mathbf{E}_\omega(\sqrt{L}).$$

Proof. Consider $h \in C_0^\infty(\mathbf{R}_+)$ such that $h(\lambda) = 1$ for $\lambda \in [0, 1]$. For a fixed $\omega > 0$ the support of $h(\lambda\omega^{-1})$ is $[0, \omega]$, which shows that for any $f \in \mathbf{E}_\omega(\sqrt{L})$ one has the equality $h(\omega^{-1}\sqrt{L})f = f$.

Applying Theorem 7.1 to the function $(\omega^{-1}\lambda)^m h(\omega^{-1}\lambda)$ we see that the operator $(\omega^{-1}\sqrt{L})^m h(\omega^{-1}\sqrt{L})$ is bounded from $L_p(\mathbf{M})$ to $L_p(\mathbf{M})$. Thus for every $f \in \mathbf{E}_\omega(\sqrt{L})$ we have

$$\|L^{m/2}f\|_p = \omega^m \left\| (\omega^{-1}\sqrt{L})^m h(\omega^{-1}L)f \right\|_p \leq C\omega^m \|f\|_p \quad \text{for } f \in \mathbf{E}_\omega(\sqrt{L}).$$

□

9.3. Besov spaces and approximations. Recall our definition of Besov spaces via

$$\mathcal{B}_{p,q}^\alpha(\mathbf{M}) = (L_p(\mathbf{M}), W_p^r(\mathbf{M}))_{\alpha/r, q}^K, \quad 0 < \alpha < r \in \mathbb{N}, \quad 1 \leq p \leq \infty, \quad 0 < q \leq \infty.$$

where K is the Peetre's interpolation functor. Note that the Sobolev space $W_p^r(\mathbf{M})$ is the domain of *any* elliptic differential operator of order r (see [143]).

Let us compare the situation on manifolds with the abstract conditions of Theorem 3.14. We treat the linear normed spaces $W_p^r(\mathbf{M})$ and $L_p(\mathbf{M})$ as the spaces E and F respectively. We identify \mathcal{T} with the linear space $\mathbf{E}_\omega(L)$ which is equipped with the quasi-norm

$$\|f\|_{\mathcal{T}} = \inf \{ \omega' : f \in \mathbf{E}_{\omega'}(L) \}, \quad f \in \mathbf{E}_\omega(L).$$

Combining Lemmas 9.1, 9.2 and Theorem 3.14 we derive the following result.

Theorem 9.3. *Fix $\alpha > 0$, $1 \leq p \leq \infty$, and $0 < q \leq \infty$. Then a function $f \in L_p(\mathbf{M})$ belongs to $\mathcal{B}_{p,q}^\alpha$ if and only if*

$$(9.3) \quad \|f\|_{\mathcal{A}_{q,p}^\alpha} := \|f\|_{L_p(\mathbf{M})} + \left(\int_0^\infty (t^\alpha \mathcal{E}(f, t, p))^q \frac{dt}{t} \right)^{1/q} < \infty.$$

Moreover,

$$(9.4) \quad \|f\|_{\mathcal{A}_{q,p}^\alpha} \sim \|f\|_{\mathcal{B}_{q,p}^\alpha}.$$

By discretizing the integral term we obtain the next theorem (see [57]).

Theorem 9.4. *Fix $\alpha > 0$, $1 \leq p \leq \infty$, and $0 < q \leq \infty$. Then a function $f \in L_p(\mathbf{M})$ belongs to $\mathcal{B}_{p,q}^\alpha$ if and only if*

$$(9.5) \quad \|f\|_{DA_{p,q}^\alpha} := \|f\|_{L_p(\mathbf{M})} + \left(\sum_{j=0}^\infty (2^{\alpha j} \mathcal{E}(f, 2^{2j}, p))^q \right)^{1/q} < \infty.$$

Moreover,

$$(9.6) \quad \|f\|_{DA_{p,q}^\alpha} \sim \|f\|_{\mathcal{B}_{p,q}^\alpha}.$$

Using this theorem and the Littlewood-Paley formula (7.6), one can easily prove the following statement (see the proof of Theorem 3.18), in which $(G_j)_{j \geq 0}$ is the family of functions defined in Subsection 3.4.

Theorem 9.5. *If $\alpha > 0$, $1 \leq p \leq \infty$, and $0 < q \leq \infty$ then $f \in \mathcal{B}_{p,q}^\alpha$ if and only if $f \in L_p(\mathbf{M})$ and*

$$(9.7) \quad \|f\|_{\tilde{A}_{p,q}^\alpha} := \|f\|_{L_p(\mathbf{M})} + \left(\sum_{j=0}^{\infty} \left(2^{\alpha j} \left\| G_j(\sqrt{L}) f \right\|_p \right)^q \right)^{1/q} < \infty.$$

Moreover,

$$(9.8) \quad \|f\|_{\tilde{A}_{p,q}^\alpha} \sim \|f\|_{\mathcal{B}_{p,q}^\alpha}.$$

9.4. Besov spaces in terms of sampling. Using the same arguments as in the proof of Theorems 5.5 and the L_p -Bernstein inequality (9.2) one can establish the following Plancherel-Polya type inequalities (for the case $p = 2$ see [104], [108]).

Theorem 9.6. *For every compact manifold \mathbf{M} there exist positive constants $c = c(\mathbf{M})$, $C_1 = C_1(\mathbf{M})$ and $C_2 = C_2(\mathbf{M})$ such that for any $\omega > 0$, every $(r, N_{\mathbf{M}})$ -lattice $\mathbf{M}_r = \{x_k\}$ with $r = c\omega^{-1}$ and every $f \in \mathbf{E}_\omega(\sqrt{L})$ the following inequalities hold true*

$$(9.9) \quad C_1 \left(\sum_k |f(x_k)|^p \right)^{1/p} \leq r^{-n/p} \|f\|_p \leq C_2 \left(\sum_k |f(x_k)|^p \right)^{1/p}.$$

Let's consider the sequence $\omega_j = 2^j$. According to Theorem 9.6 there exists a $c > 0$, $C_1 > 0$, $C_2 > 0$ such that for any $(r_j, N_{\mathbf{M}})$ -lattice $\mathbf{M}_{r_j} = \{x_{j,k}\}$ with

$$r_j = c\omega_j^{-1} = c2^{-j}.$$

the Plancherel-Polya inequality (9.9) holds for every $f \in \mathbf{E}_{\omega_j}(\sqrt{L})$. It implies another characterization of Besov spaces (see [118] for the case $p = 2$). Again, $(G_j)_{j \geq 0}$ is as defined in Subsection 3.4.

Theorem 9.7. *Given $f \in L_p(\mathbf{M})$, $\alpha > 0$, $1 \leq p \leq \infty$, and $0 < q \leq \infty$. Then $f \in \mathcal{B}_{p,q}^\alpha$ if and only if*

$$(9.10) \quad \|f\|_{\tilde{A}_{p,q}^\alpha} := \|f\|_{L_p(\mathbf{M})} + \left(\sum_{j=0}^{\infty} 2^{jq(\alpha-n/p)} \left(\sum_k |G_j(\sqrt{L}) f(x_{j,k})|^p \right)^{q/p} \right)^{1/q} < \infty.$$

Moreover,

$$(9.11) \quad \|f\|_{\tilde{A}_{p,q}^\alpha} \sim \|f\|_{\mathcal{B}_{p,q}^\alpha}.$$

Proof. By construction every function $G_j(\sqrt{L})f$ belongs to $\mathbf{E}_{\omega_j}(\sqrt{L})$. According to (9.9) one has for every $f \in L_p$

$$C_1 2^{-jn/p} \left(\sum_k |G_j(\sqrt{L}) f(x_{j,k})|^p \right)^{1/p} \leq \|G_j(\sqrt{L}) f\|_p \leq$$

$$(9.12) \quad C_2 2^{-jn/p} \left(\sum_k \left| G_j(\sqrt{L}) f(x_{j,k}) \right|^p \right)^{1/p}.$$

Using Theorem 9.5 we obtain the statement. This concludes the proof of the theorem. \square

10. APPROXIMATION THEORY, BESOV SPACES ON COMPACT HOMOGENEOUS MANIFOLDS

10.1. Bernstein spaces on compact homogeneous manifolds. For detailed proofs of all the statements in this subsection see [98], [99], [115]. Returning to the compact homogeneous manifold $\mathbf{M} = G/K$, let $\mathbb{D} = \{D_1, \dots, D_d\}$, $d = \dim G$, be the same set of operators as in (4.12), and $\mathcal{L} = -D_1^2 - \dots - D_d^2$. Let us define the *Bernstein space*

$$\mathbf{B}_\omega^\mathbf{p}(\mathbb{D}) = \{f \in L_p(\mathbf{M}) : \|D_{i_1} \dots D_{i_m} f\|_p \leq \omega^m \|f\|_p, \ 1 \leq i_1, \dots, i_m \leq d, \ \omega \geq 0\},$$

where $d = \dim G$.

As before, the notation $\mathbf{E}_\omega(\sqrt{\mathcal{L}})$, $\omega \geq 0$, will be used for a span of eigenvectors of $\sqrt{\mathcal{L}}$ with eigenvalues $\leq \omega$. For these spaces the next two theorems hold (see [99], [115]):

Theorem 10.1. *The following properties hold:*

(1)

$$\mathbf{B}_\omega^\mathbf{p}(\mathbb{D}) = \mathbf{B}_\omega^\mathbf{q}(\mathbb{D}), \quad 1 \leq p \leq q \leq \infty, \ \omega \geq 0.$$

(2)

$$\mathbf{B}_\omega^\mathbf{p}(\mathbb{D}) \subset \mathbf{E}_{\omega^{2d}}(\sqrt{\mathcal{L}}) \subset \mathbf{B}_{\omega\sqrt{d}}^\mathbf{p}(\mathbb{D}), \quad d = \dim G, \ \omega \geq 0.$$

(3) *(Bernstein-Nikolskii inequality)*

$$(10.1) \quad \|\mathcal{L}^m \varphi\|_q \leq C(\mathbf{M}) \omega^{2m + \frac{d}{p} - \frac{d}{q}} \|\varphi\|_p, \quad \varphi \in \mathbf{E}_\omega(\sqrt{\mathcal{L}}), \quad m \in \mathbb{N},$$

where $d = \dim G$, $1 \leq p \leq q \leq \infty$.

Remark 10.2. *When $p = q$ the inequality (10.1) becomes*

$$(10.2) \quad \|\mathcal{L}^m \varphi\|_p \leq C(\mathbf{M}) \omega^{2m} \|\varphi\|_p, \quad \varphi \in \mathbf{E}_\omega(\sqrt{\mathcal{L}}), \quad m \in \mathbb{N}.$$

Note that the inequality (9.2) is weaker than the inequality (10.2) in the sense that the constant in (9.2) depends on m (and obviously on the manifold) but the constant in (10.2) depends only on the manifold.

Every compact Lie group can be considered to be a closed subgroup of the orthogonal group $O(\mathbb{R}^N)$ of some Euclidean space \mathbb{R}^N . It means that we can identify $\mathbf{M} = G/K$ with the orbit of a unit vector $v \in \mathbb{R}^N$ under the action of a subgroup of the orthogonal group $O(\mathbb{R}^N)$ in some \mathbb{R}^N . In this case K will be the stationary group of v . Such an embedding of \mathbf{M} into \mathbb{R}^N is called *equivariant*.

We choose an orthonormal basis in \mathbb{R}^N for which the first vector is the vector v : $e_1 = v, e_2, \dots, e_N$. Let $\mathbf{P}_r(\mathbf{M})$ be the space of restrictions to \mathbf{M} of all polynomials in \mathbb{R}^N of degree r . This space is closed in the norm of $L_p(\mathbf{M})$, $1 \leq p \leq \infty$, which is constructed with respect to the G -invariant normalized measure on \mathbf{M} .

Theorem 10.3. *If \mathbf{M} is embedded into an \mathbb{R}^N equivariantly, then*

$$P_r(\mathbf{M}) \subset B_r(\mathbb{D}) \subset E_{r^2d}(\sqrt{\mathcal{L}}) \subset B_{r\sqrt{d}}(\mathbb{D}), \quad d = \dim G, \quad r \in \mathbb{N},$$

and

$$\text{span}_{r \in \mathbb{N}} P_r(\mathbf{M}) = \text{span}_{\omega \geq 0} B_\omega(\mathbb{D}) = \text{span}_{j \in \mathbb{N}} E_{\lambda_j}(\sqrt{\mathcal{L}}).$$

10.2. Mixed modulus of continuity and Besov spaces on compact homogeneous manifolds. For more details on the topic of this section see [94], [95]. For the same operators as above D_1, \dots, D_d , $d = \dim G$, (see Section 3) let T_1, \dots, T_d be the corresponding one-parameter groups of translation along integral curves of the corresponding vector fields i.e.

$$(10.3) \quad T_j(\tau)f(x) = f(\exp \tau X_j \cdot x), \quad x \in \mathbf{M} = G/K, \quad \tau \in \mathbb{R}, \quad f \in L_p(\mathbf{M}), \quad 1 \leq p < \infty,$$

here $\exp \tau X_j \cdot x$ is the integral curve of the vector field X_j which passes through the point $x \in \mathbf{M}$. The modulus of continuity is introduced as

$$\Omega_p^r(s, f) =$$

$$(10.4) \quad \sum_{1 \leq j_1, \dots, j_r \leq d} \sup_{0 \leq \tau_{j_1} \leq s} \dots \sup_{0 \leq \tau_{j_r} \leq s} \| (T_{j_1}(\tau_{j_1}) - I) \dots (T_{j_r}(\tau_{j_r}) - I) f \|_{L_p(\mathbf{M})},$$

where $d = \dim G$, $f \in L_p(\mathbf{M})$, $1 \leq p < \infty$, $r \in \mathbb{N}$, and I is the identity operator in $L_p(\mathbf{M})$. We consider the space of all functions in $L_p(\mathbf{M})$ for which the following norm is finite:

$$(10.5) \quad \|f\|_{L_p(\mathbf{M})} + \left(\int_0^\infty (s^{-\alpha} \Omega_p^r(s, f))^q \frac{ds}{s} \right)^{1/q}, \quad 1 \leq p, q < \infty,$$

with the usual modifications for $q = \infty$.

It is known [145]-[147] that the Besov space $\mathcal{B}_{p,q}^\alpha(\mathbf{M})$ are exactly the interpolation space according to Peetre's K-method:

$$\mathcal{B}_{p,q}^\alpha(\mathbf{M}) = (L_p(\mathbf{M}), W_p^r(\mathbf{M}))_{\alpha/r, q}^K, \quad 0 < \alpha < r \in \mathbb{N}, \quad 1 \leq p \leq \infty, \quad 0 < q \leq \infty.$$

where K is the Peetre interpolation functor.

The following theorem follows from a more general results in [94]-[98].

Theorem 10.4. *If $\mathbf{M} = G/K$ is a compact homogeneous manifold the norm of the Besov space $\mathcal{B}_{p,q}^\alpha(\mathbf{M})$, $0 < \alpha < r \in \mathbb{N}$, $1 \leq p, q < \infty$, is equivalent to the norm (10.5). Moreover, with $d = \dim G$ the norm in (10.5) is equivalent to the norm*

$$(10.6) \quad \|f\|_{W_p^{[\alpha]}(\mathbf{M})} + \sum_{1 \leq j_1, \dots, j_{[\alpha]} \leq d} \left(\int_0^\infty \left(s^{[\alpha]-\alpha} \Omega_p^1(s, D_{j_1} \dots D_{j_{[\alpha]}} f) \right)^q \frac{ds}{s} \right)^{1/q},$$

if α is not an integer ($[\alpha]$ is its integer part). For the integer case $\alpha = k \in \mathbb{N}$ the norm (10.5) is equivalent to the norm (Zygmund condition)

$$(10.7) \quad \|f\|_{W_p^{k-1}(\mathbf{M})} + \sum_{1 \leq j_1, \dots, j_{k-1} \leq d} \left(\int_0^\infty (s^{-1} \Omega_p^2(s, D_{j_1} \dots D_{j_{k-1}} f))^q \frac{ds}{s} \right)^{1/q}.$$

10.3. Besov spaces in terms of the frame coefficients. Let us note that for the frame functions Θ_k^j which were introduced in (8.3) the inequalities (7.1) and (8.5) imply that there exists a constant $C > 0$ such that uniformly in j and k the following estimate holds

$$(10.8) \quad \|\Theta_k^j\|_p \leq C 2^{nj(1/2-1/p)}.$$

This estimate can be improved. Indeed, the following improvement on Corollary 7.1 for compact manifolds holds true [59].

Theorem 10.5. *Given a compact manifold \mathbf{M} and an $F \in C_0^\infty(\mathbf{R})$, one has the following asymptotic behavior for any $1 \leq p \leq \infty$:*

$$(10.9) \quad \left(\int_{\mathbf{M}} |K_t^F(x, y)|^p dy \right)^{1/p} \asymp t^{-n/q}, \text{ for } t \rightarrow 0 \quad 1/p + 1/q = 1, \quad 1 \leq p \leq \infty,$$

with constants independent of x and t , as $t \rightarrow 0$.

By applying this theorem to the frame functions Θ_k^j we obtain the following improvement of the estimate (10.8)

$$(10.10) \quad \left\| \Theta_k^j \right\|_p \asymp 2^{nj(1/2-1/p)}, \quad 1 \leq p \leq \infty.$$

The quasi-Banach space $\mathbf{b}_{p,q}^\alpha$ consists of sequences $s = \{s_k^j\}$ ($j \geq 0, 1 \leq k \leq \mathcal{K}_j$) satisfying

$$(10.11) \quad \|s\|_{\mathbf{b}_{p,q}^\alpha} = \left(\sum_{j \geq 0} 2^{jq(\alpha-n/p+n/2)} \left(\sum_k |s_k^j|^p \right)^{q/p} \right)^{1/q} < \infty.$$

We consider the following mappings

$$(10.12) \quad \tau(f) = \{\langle f, \Theta_k^j \rangle\},$$

and

$$(10.13) \quad \sigma(\{s_k^j\}) = \sum_{j \geq 0} \sum_k s_k^j \Theta_k^j,$$

defined on the space of finitely supported coefficient sequences. By using the relation (10.10) one can prove the following theorem which appeared in [57] and which characterizes Besov spaces on \mathbf{M} in terms of the frame coefficients.

Theorem 10.6. *Let Θ_k^j be given as in (8.3). Then for $1 \leq p \leq \infty, 0 < q \leq \infty, \alpha > 0$ the following statements are valid:*

- (1) τ in (10.12) is a well defined bounded operator $\tau : \mathcal{B}_{p,q}^\alpha(\mathbf{M}) \rightarrow \mathbf{b}_{p,q}^\alpha$;
- (2) σ in (10.13) is a well defined bounded operator $\sigma : \mathbf{b}_{p,q}^\alpha \rightarrow \mathcal{B}_{p,q}^\alpha(\mathbf{M})$;
- (3) $\sigma \circ \tau = id$;
- (4) the following norms are equivalent:

$$\|f\|_{\mathcal{B}_{p,q}^\alpha(\mathbf{M})} \asymp \left(\sum_{j=0}^{\infty} 2^{jq(\alpha-n/p+n/2)} \left(\sum_k |\langle f, \Theta_k^j \rangle|^p \right)^{q/p} \right)^{1/q} = \|\tau(f)\|_{\mathbf{b}_{p,q}^\alpha}.$$

Moreover, the constants in these norm equivalence relations can be estimated uniformly over compact ranges of the parameters p, q, α .

In fact, the frame expansions obtained in the Hilbert space setting extend to Banach frames for the corresponding family of Besov spaces, a situation which is quite well known from coorbit theory (see [60]).

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